CAMBRIDGE INTERNATIONAL EXAMINATIONS
General Certificate of Education
Advanced Level

FURTHER MATHEMATICS **9231/01**

Paper 1

May/June 2003

3 hours

Additional materials: Answer Booklet/Paper
Graph paper
List of Formulae (MF10)

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

This document consists of **5** printed pages and **3** blank pages.

UNIVERSITY of CAMBRIDGE
Local Examinations Syndicate

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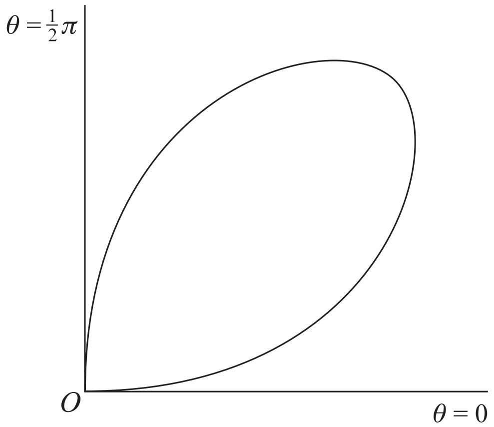

The diagram shows one loop of the curve whose polar equation is $r = a \sin 2\theta$, where a is a positive constant. Find the area of the loop, giving your answer in terms of a and π. [4]

2 Prove by induction that, for all $N \geqslant 1$,

$$\sum_{n=1}^{N} \frac{n+2}{n(n+1)2^n} = 1 - \frac{1}{(N+1)2^N}.$$ [5]

3 Let $v_1, v_2, v_3, \ldots$ be a sequence and let

$$u_n = nv_n - (n+1)v_{n+1},$$

for $n = 1, 2, 3, \ldots$. Find $\sum_{n=1}^{N} u_n$. [2]

In each of the following cases determine whether the series $u_1 + u_2 + u_3 + \ldots$ is convergent, and justify your conclusion. Give the sum to infinity where this exists.

(i) $v_n = n^{-\frac{1}{2}}$. [2]

(ii) $v_n = n^{-\frac{3}{2}}$. [2]

4 The curve C has equation $y = \dfrac{x^2 - 4}{x - 3}$.

(i) Find the equations of the asymptotes of C. [3]

(ii) Draw a sketch of C and its asymptotes. Give the coordinates of the points of intersection of C with the coordinate axes. [4]

[You are not required to find the coordinates of any turning points.]

5 The equation

$$8x^3 + 12x^2 + 4x - 1 = 0$$

has roots α, β, γ. Show that the equation with roots $2\alpha + 1, 2\beta + 1, 2\gamma + 1$ is

$$y^3 - y - 1 = 0. \qquad [3]$$

The sum $(2\alpha + 1)^n + (2\beta + 1)^n + (2\gamma + 1)^n$ is denoted by S_n. Find the values of S_3 and S_{-2}. [5]

6 Use de Moivre's theorem to show that

$$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1. \qquad [5]$$

Hence solve the equation

$$64x^6 - 96x^4 + 36x^2 - 1 = 0,$$

giving each root in the form $\cos k\pi$. [4]

7 The variables x and y are related by the equation $x^4 + y^4 = 1$, where $0 < x < 1$ and $0 < y < 1$.

(i) Obtain an equation which relates $x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$, and deduce that

$$\frac{d^2y}{dx^2} = -\frac{3x^2}{y^7}. \qquad [6]$$

(ii) Given that $y = b_1$ when $x = a_1$ and that $y = b_2$ when $x = a_2$, where $a_1 < a_2$, prove that the mean value of $\frac{d^3y}{dx^3}$ with respect to x over the interval $a_1 \leqslant x \leqslant a_2$ is

$$\frac{3(a_1^2 b_2^7 - a_2^2 b_1^7)}{b_1^7 b_2^7 (a_2 - a_1)}. \qquad [4]$$

8 The linear transformation $\mathrm{T} : \mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -2 & 3 \\ 2 & -1 & -1 & 11 \\ 3 & -2 & -3 & 14 \\ 4 & -3 & -5 & 17 \end{pmatrix}.$$

Find the rank of $\mathbf{A}$ and a basis for the null space of T. [7]

The vector $\begin{pmatrix} 1 \\ -2 \\ -1 \\ -1 \end{pmatrix}$ is denoted by $\mathbf{e}$. Show that there is a solution of the equation $\mathbf{Ax} = \mathbf{Ae}$ of the form $\mathbf{x} = \begin{pmatrix} p \\ q \\ 1 \\ 1 \end{pmatrix}$, where p and q are to be found. [4]

9 The variables x and t, where $x > 0$ and $0 \leqslant t \leqslant \frac{1}{2}\pi$, are related by

$$x\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + 5x\frac{\mathrm{d}x}{\mathrm{d}t} + 3x^2 = 3\sin 2t + 15\cos 2t,$$

and the variables x and y are related by $y = x^2$. Show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 5\frac{\mathrm{d}y}{\mathrm{d}t} + 6y = 6\sin 2t + 30\cos 2t. \qquad [3]$$

Hence find x in terms of t, given that $x = 2$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{3}{2}$ when $t = 0$. [10]

10 Find the acute angle between the planes with equations

$$x - 2y + z - 9 = 0 \qquad \text{and} \qquad x + y - z + 2 = 0. \qquad [3]$$

The planes meet in the line l, and A is the point on l whose position vector is $p\mathbf{i} + q\mathbf{j} + \mathbf{k}$.

(i) Find p and q. [2]

(ii) Find a vector equation for l. [3]

The non-coincident planes Π_1 and Π_2 are both perpendicular to l. The perpendicular distance from A to Π_1 is $\sqrt{14}$ and the perpendicular distance from A to Π_2 is also $\sqrt{14}$. Find equations for Π_1 and Π_2 in the form $ax + by + cz = d$. [5]

11 Answer only **one** of the following two alternatives.

EITHER

Given that

$$I_n = \int_0^1 x^n e^{-\alpha x}\, dx,$$

where α is a positive constant and n is a non-negative integer, show that for $n \geqslant 1$,

$$\alpha I_n = nI_{n-1} - e^{-\alpha}. \qquad [3]$$

Hence, or otherwise, find the coordinates of the centroid of the finite region bounded by the x-axis, the line $x = 1$ and the curve $y = xe^{-x}$, giving your answers in terms of e. [11]

OR

The vector **e** is an eigenvector of each of the $n \times n$ matrices **A** and **B**, with corresponding eigenvalues λ and μ respectively. Prove that **e** is an eigenvector of the matrix **AB** with eigenvalue $\lambda\mu$. [3]

Find the eigenvalues and corresponding eigenvectors of the matrix **C**, where

$$\mathbf{C} = \begin{pmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 2 & 1 & 2 \end{pmatrix}. \qquad [8]$$

Verify that one of the eigenvectors of **C** is an eigenvector of the matrix **D**, where

$$\mathbf{D} = \begin{pmatrix} -3 & 1 & 1 \\ 0 & -2 & 4 \\ 0 & 0 & -4 \end{pmatrix}. \qquad [2]$$

Hence find an eigenvalue of the matrix **CD**. [1]

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