UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
General Certificate of Education Advanced Level

FURTHER MATHEMATICS **9231/01**

Paper 1 **October/November 2009**

3 hours

Additional Materials: Answer Booklet/Paper
Graph Paper
List of Formulae (MF10)

0307464767

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **5** printed pages and **3** blank pages.

1 Given that

$$y = x^2 \sin x,$$

(i) show that the mean value of $\frac{dy}{dx}$ with respect to x over the interval $0 \leqslant x \leqslant \frac{1}{2}\pi$ is $\frac{1}{2}\pi$, [2]

(ii) find the mean value of $\frac{d^2y}{dx^2}$ with respect to x over the interval $0 \leqslant x \leqslant \frac{1}{2}\pi$. [2]

2 Relative to an origin O, the points A, B, C have position vectors

$$\mathbf{i}, \quad \mathbf{j} + \mathbf{k}, \quad \mathbf{i} + \mathbf{j} + \theta\mathbf{k},$$

respectively. The shortest distance between the lines AB and OC is $\frac{1}{\sqrt{2}}$. Find the value of θ. [6]

3 The curve C has equation

$$y = \frac{x^2 - 5x + 4}{x + 1}.$$

(i) Obtain the coordinates of the points of intersection of C with the axes. [2]

(ii) Obtain the equation of each of the asymptotes of C. [3]

(iii) Draw a sketch of C. [3]

4 It is given that

$$x = t + \sin t, \quad y = t^2 + 2\cos t,$$

where $-\pi < t < \pi$. Find $\frac{dy}{dx}$ in terms of t. [2]

Show that

$$\frac{d^2y}{dx^2} = \frac{2t \sin t}{(1 + \cos t)^3}.$$ [4]

Show that $\frac{dy}{dx}$ increases with x over the given interval of t. [2]

5 The equation

$$x^3 + 5x + 3 = 0$$

has roots α, β, γ. Use the substitution $x = -\frac{3}{y}$ to find a cubic equation in y and show that the roots of this equation are $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$. [4]

Find the exact values of $\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2$ and $\beta^3\gamma^3 + \gamma^3\alpha^3 + \alpha^3\beta^3$. [5]

6 Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{n-1}\sqrt{(4-x^2)}\right] = \frac{4(n-1)x^{n-2}}{\sqrt{(4-x^2)}} - \frac{nx^n}{\sqrt{(4-x^2)}}. \qquad [3]$$

Let

$$I_n = \int_0^1 \frac{x^n}{\sqrt{(4-x^2)}}\,\mathrm{d}x,$$

where $n \geqslant 0$. Prove that

$$nI_n = 4(n-1)I_{n-2} - \sqrt{3},$$

for $n \geqslant 2$. [2]

Given that $I_0 = \frac{1}{6}\pi$, find I_4, leaving your answer in an exact form. [4]

7 Use de Moivre's theorem to express $\sin^6\theta$ in the form

$$a + b\cos 2\theta + c\cos 4\theta + d\cos 6\theta,$$

where a, b, c, d are constants to be found. [5]

Hence evaluate

$$\int_0^{\frac{1}{4}\pi} \sin^6 2x\,\mathrm{d}x,$$

leaving your answer in terms of π. [4]

8 **(a)** The curve C_1 has equation $y = -\ln(\cos x)$. Show that the length of the arc of C_1 from the point where $x = 0$ to the point where $x = \frac{1}{3}\pi$ is $\ln(2 + \sqrt{3})$. [5]

(b) The curve C_2 has equation $y = 2\sqrt{(x+3)}$. The arc of C_2 joining the point where $x = 0$ to the point where $x = 1$ is rotated through one complete revolution about the x-axis. Show that the area of the surface generated is

$$\frac{8}{3}\pi(5\sqrt{5} - 8). \qquad [5]$$

9 Show that if y depends on x and $x = e^u$ then

$$x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}. \qquad [4]$$

Given that y satisfies the differential equation

$$x^2 \frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 3y = 30x^2,$$

use the substitution $x = e^u$ to show that

$$\frac{d^2y}{du^2} + 4\frac{dy}{du} + 3y = 30e^{2u}. \qquad [2]$$

Hence find the general solution for y in terms of x. [5]

10 The curve C has polar equation

$$r = a \sin 3\theta,$$

where $0 \leqslant \theta \leqslant \frac{1}{3}\pi$.

(i) Show that the area of the region enclosed by C is $\frac{1}{12}\pi a^2$. [3]

(ii) Show that, at the point of C at maximum distance from the initial line,

$$\tan 3\theta + 3\tan\theta = 0. \qquad [3]$$

(iii) Use the formula

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

to find this maximum distance. [4]

(iv) Draw a sketch of C. [2]

11 Answer only **one** of the following two alternatives.

EITHER

Prove by induction that

$$\sum_{n=1}^{N} n^3 = \tfrac{1}{4}N^2(N+1)^2. \qquad [5]$$

Use this result, together with the formula for $\sum_{n=1}^{N} n^2$, to show that

$$\sum_{n=1}^{N}(20n^3 + 36n^2) = N(N+1)(N+3)(5N+2). \qquad [3]$$

Let

$$S_N = \sum_{n=1}^{N}(20n^3 + 36n^2 + \mu n).$$

Find the value of the constant μ such that S_N is of the form $N^2(N+1)(aN+b)$, where the constants a and b are to be determined. [3]

Show that, for this value of μ,

$$5 + \frac{22}{N} < N^{-4}S_N < 5 + \frac{23}{N},$$

for all $N \geqslant 18$. [3]

OR

One of the eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -4 & 6 \\ 2 & -4 & 2 \\ -3 & 4 & a \end{pmatrix}$$

is -2. Find the value of a. [3]

Another eigenvalue of $\mathbf{A}$ is -5. Find eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$ corresponding to the eigenvalues -2 and -5 respectively. [3]

The linear space spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$ is denoted by V.

(i) Prove that, for any vector $\mathbf{x}$ belonging to V, the vector $\mathbf{Ax}$ also belongs to V. [4]

(ii) Find a non-zero vector which is perpendicular to every vector in V, and determine whether it is an eigenvector of $\mathbf{A}$. [4]

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