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**FURTHER MATHEMATICS**

**9231/01**

Paper 1

**For examination from 2017**

MARK SCHEME

Maximum Mark: 100

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**Specimen**

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This document consists of **16** printed pages.

**Mark Scheme Notes**

Marks are of the following three types:

- M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.
- A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).
- B Mark for a correct result or statement independent of method marks.
- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep\*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.
  - The symbol  $\surd$  implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.
  - Note: B2 or A2 means that the candidate can earn 2 or 0.  
B2/1/0 means that the candidate can earn anything from 0 to 2.

The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.
- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking  $g$  equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

- AEF Any Equivalent Form (of answer is equally acceptable)
- AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
- BOD Benefit of Doubt (allowed when the validity of a solution may not be absolutely clear)
- CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)
- CWO Correct Working Only – often written by a 'fortuitous' answer
- ISW Ignore Subsequent Working
- MR Misread
- PA Premature Approximation (resulting in basically correct work that is insufficiently accurate)
- SOS See Other Solution (the candidate makes a better attempt at the same question)
- SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

### **Penalties**

- MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through  $\checkmark$ " marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR–2 penalty may be applied in particular cases if agreed at the coordination meeting.
- PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

Question	Answer	Marks	Partial Marks	Guidance
<b>1</b>	$\dot{x} = -6\cos^2 t \sin t, \dot{y} = 6\sin^2 t \cos t$	1	<b>B1</b>	
	$\Rightarrow \frac{dy}{dx} = -\tan t$ (OE)	1	<b>B1</b>	
	$\frac{d^2y}{dx^2} = -\sec^2 t \times \frac{-1}{6\cos^2 t \sin t} = \frac{1}{6}\sec^4 t \operatorname{cosec} t$ AG	2	<b>M1A1</b>	
		<b>4</b>		
<b>2</b>	$m^2 + 4m + 4 = 0 \Rightarrow (m+2)^2 = 0 \Rightarrow m = -2$	1	<b>M1</b>	
	CF: $Ae^{-2t} + Bte^{-2t}$ soi	1	<b>A1</b>	
	PI: $x = pt^2 + qt + r \Rightarrow \dot{x} = 2pt + q \Rightarrow \ddot{x} = 2p$	1	<b>M1</b>	
	$\Rightarrow 2p + 8pt + 4q + 4pt^2 + 4qt + 4r = 7 - 2t^2$	1	<b>M1</b>	
	$\Rightarrow p = -\frac{1}{2}, q = 1, r = 1$	1	<b>A1</b>	
	GS: $x = Ae^{-2t} + Bte^{-2t} - \frac{1}{2}t^2 + t + 1$	1	<b>A1</b>	
		<b>6</b>		

Question	Answer	Marks	Partial Marks	Guidance
<b>3</b>	$n = 1$ in formula gives $a^0 e^{ax} + axe^{ax} = e^{ax} + axe^{ax}$	1	<b>B1</b>	
	$\frac{d}{dx}(xe^{ax}) = e^{ax} \times 1 + x \cdot ae^{ax} = e^{ax} + axe^{ax} \Rightarrow H_1$ is true oe	1	<b>B1</b>	
	Assume $H_k$ is true, i.e. $\frac{d^k}{dx^k}(xe^{ax}) = ka^{k-1}e^{ax} + a^k xe^{ax}$ .	1	<b>B1</b>	
	$\frac{d^{k+1}}{dx^{k+1}}(xe^{ax}) = ka^k e^{ax} + a^k e^{ax} + a^{k+1} xe^{ax}$	1	<b>M1</b>	
	$= (k+1)a^k e^{ax} + a^{k+1} xe^{ax}$	1	<b>A1</b>	
	$\Rightarrow H_{k+1}$ is true, hence by PMI $H_n$ is true for all positive integers $n$ .	1	<b>A1</b>	
		<b>6</b>		
<b>4(i)</b>	$\left(\frac{6}{\sqrt{1}} - \frac{7}{\sqrt{3}}\right) + \left(\frac{7}{\sqrt{3}} - \frac{8}{\sqrt{7}}\right) + \dots + \left(\frac{35}{\sqrt{871}} - \frac{36}{\sqrt{931}}\right) = 6 - \frac{36}{\sqrt{931}} = 4.820$	<b>3</b>	<b>M1A1A1</b>	
<b>4(ii)</b>	$6 - \frac{n+6}{\sqrt{n^2+n+1}} > 4.9 \Rightarrow 0.21n^2 - 10.79n - 34.79 (> 0)$	2	<b>M1*A1</b>	
	$\Rightarrow n > 54.42\dots$ so 55 terms required.	2	<b>DM1A1</b>	
		<b>4</b>		

Question	Answer	Marks	Partial Marks	Guidance
5	$\alpha + \beta + \gamma = -p = 15 \Rightarrow p = -15$	1	<b>B1</b>	
	$2(\alpha\beta + \beta\gamma + \gamma\alpha) = (\alpha + \beta + \gamma)^2 - (\alpha^2 + \beta^2 + \gamma^2) = 2q$	1	<b>M1</b>	
	$\Rightarrow q = \frac{1}{2}(225 - 83) = 71$	1	<b>A1</b>	
		<b>3</b>		
	$\frac{36}{\alpha} = 15 - \alpha \quad (= [\beta + \gamma])$	1	<b>M1</b>	
	$\Rightarrow a^2 - 15\alpha + 36 = 0 \Rightarrow \alpha = 3, \alpha \neq 12, \text{ e.g. since } 12^2 > 83 \text{ or other reason}$	2	<b>M1A1</b>	
	$\beta\gamma = 71 - 36 = 35$	1	<b>B1</b>	
	$\Rightarrow r = -\alpha\beta\gamma = -3 \times 35 = -105$	1	<b>A1</b>	(extra answer penalised)
	<b>5</b>			

Question	Answer	Marks	Partial Marks	Guidance
6	$\lambda = 1: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 10 & -8 & 10 \\ 7 & -5 & 7 \end{vmatrix} = \begin{pmatrix} -6 \\ 0 \\ 6 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ oe	2	M1A1	
	$\lambda = 3: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & 0 \\ 10 & -10 & 10 \end{vmatrix} = \begin{pmatrix} 0 \\ 20 \\ 20 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ oe	1	A1	
		3		
	$\begin{pmatrix} 1 & 0 & 0 \\ 10 & -7 & 10 \\ 7 & -5 & 8 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \\ -2 \end{pmatrix} = -2 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \lambda = -2$	2	M1A1	
	$\mathbf{D} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ , $\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}$ (or other multiples or permutations).	2	B1 <sup>✓</sup> B1 <sup>✓</sup>	
	Det $\mathbf{P} = -1$ (or 1 depending on permutation).	1	B1	
	Adj $\mathbf{P} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ -2 & 1 & 2 \end{pmatrix} \Rightarrow \mathbf{P}^{-1} = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 0 & 0 \\ 2 & -1 & 2 \end{pmatrix}$ (or other permutations).	2	M1A1	
	5			

Question	Answer	Marks	Partial Marks	Guidance
7	$\begin{pmatrix} 1 & -2 & -3 & 1 \\ 3 & -5 & -7 & 7 \\ 5 & -9 & -13 & 9 \\ 7 & -13 & -19 & 11 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & -3 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & -3 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	2	M1A1	
	$r(\mathbf{M}) = 4 - 2 = 2$	1	A1	
	$\begin{aligned} x - 2y - 3z + t &= 0 \\ y + 2z + 4t &= 0 \end{aligned}$	1	M1	
	E.g. Set $z = \lambda$ and $t = \mu \Rightarrow y = -2\lambda - 4\mu$ and $x = -\lambda - 9\mu$	1	M1	
	$\Rightarrow \text{Basis } \left\{ \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -9 \\ -4 \\ 0 \\ 1 \end{pmatrix} \right\}$	1	A1	
		6		
	$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -9 \\ -4 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ -1 \\ -1 \end{pmatrix}$	1	M1	
	Solving: $\lambda = -4$ and $\mu = -5$ $\Rightarrow a = 50, b = 30.$	3	M1 A1 A1	
		4		



Question	Answer	Marks	Partial Marks	Guidance
8	$y' = 0 \Rightarrow (x+1)(4x+k) - (2x^2+kx) \times 1 = 0$	1	<b>M1</b>	
	$\Rightarrow 4x^2 + (4+k)x + k - 2x^2 - kx = 0 \Rightarrow 2x^2 + 4x + k = 0$	1	<b>A1</b>	
	$B^2 - 4AC < 0 \Rightarrow$ no stationary points $\Rightarrow 16 - 8k < 0$ $\Rightarrow k > 2$ for no stationary points.	3	<b>M1A1</b> <b>A1</b>	
		<b>5</b>		
	When $k = 4$ : Vertical asymptote: $x = -1$	1	<b>B1</b>	
	Oblique asymptote: $y = 2x + 2 - \frac{2}{x+1} \Rightarrow y = 2x + 2$	2	<b>M1A1</b>	
	Axes and asymptotes Each branch.	3	<b>B1</b> <b>B1B1</b>	
		<b>6</b>		

Question	Answer	Marks	Partial Marks	Guidance
9	$\int_1^e \ln x dx = x \ln x - x$	1	<b>B1</b>	
	$I_n = \int_1^e (\ln x)^{n-1} \cdot \ln x dx$	1	<b>M1</b>	
	$= \left[ (\ln x)^{n-1} (x \ln x - x) \right]_1^e - \int_1^e (n-1)(\ln x)^{n-2} \cdot \frac{1}{x} (x \ln x - x) dx$	2	<b>M1A1</b>	
	$= 0 - \int_1^e (n-1)(\ln x)^{n-2} (\ln x - 1) dx = (n-1)[I_{n-2} - I_{n-1}] \quad (\text{AG})$	2	<b>M1A1</b>	
	<b>Alternative for obtaining reduction formula:</b>			
	$I_n = \int_1^e (\ln x)^n \times 1 dx = \left[ x(\ln x)^n \right]_1^e - \int_1^e n(\ln x)^{n-1} dx$	2	<b>M1A1</b>	
	$\Rightarrow I_n = e - nI_{n-1}$	1	<b>A1</b>	
	Similarly $I_{n-1} = e - (n-1)I_{n-2}$	1	<b>B1</b>	
	$\Rightarrow I_n + nI_{n-1} = I_{n-1} + (n-1)I_{n-2}$	1	<b>M1</b>	
$\Rightarrow I_n = (n-1)[I_{n-2} - I_{n-1}] \quad (\text{AG})$	1	<b>A1</b>		
		<b>6</b>		

Question	Answer	Marks	Partial Marks	Guidance
	$I_0 = [x]_1^e = e - 1$	1	<b>B1</b>	
	$I_1 = [x \ln x - x]_1^e = 1$	1	<b>B1</b>	
	$I_2 = 1 \times (e - 1 - 1) = e - 2$	1	<b>M1</b>	
	$I_3 = 2(I_1 - I_2) = 2(1 - [e - 2]) = 6 - 2e$	1	<b>A1</b>	
	$MV = \frac{I_3}{e-1} = \frac{6-2e}{e-1}$	2	<b>M1 A1<sup>✓</sup></b>	
		<b>6</b>		

Question	Answer	Marks	Partial Marks	Guidance
<b>10</b>	$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$	1	<b>B1</b>	
	$(c + is)^5 = c^5 + 5c^4si - 10c^3s^2 - 10ic^2s^3i + 5cs^4 + s^5i$	2	<b>M1A1</b>	
	$\tan 5\theta = \frac{5c^4s - 10c^2s^3 + s^5}{c^5 - 10c^3s^2 + 5cs^4}$	1	<b>M1</b>	
	Divide numerator and denominator by $c^5$ (stated or shown): $\Rightarrow \tan 5\theta = \frac{5 \tan \theta - 10 \tan^3 \theta + \tan^5 \theta}{1 - 10 \tan^2 \theta + 5 \tan^4 \theta}$ (AG)	1	<b>A1</b>	
		<b>5</b>		

Question	Answer	Marks	Partial Marks	Guidance
	$\tan 5\theta = 0 \Rightarrow \theta = \frac{1}{5}\pi, \frac{2}{5}\pi, \frac{3}{5}\pi, \frac{4}{5}\pi, \pi$	1	<b>B1</b>	
	$t^5 - 10t^3 + 5t = 0$ has roots $\tan\left(\frac{1}{5}\pi\right), \tan\left(\frac{2}{5}\pi\right), \tan\left(\frac{3}{5}\pi\right), \tan\left(\frac{4}{5}\pi\right), \tan \pi$ $\Rightarrow t^4 - 10t^2 + 5 = 0$ has roots $\tan\left(\frac{1}{5}\pi\right), \tan\left(\frac{2}{5}\pi\right), \tan\left(\frac{3}{5}\pi\right), \tan\left(\frac{4}{5}\pi\right)$ .	1	<b>B1</b>	
	$\Rightarrow \left(t^2 - \tan^2\left(\frac{1}{5}\pi\right)\right)\left(t^2 - \tan^2\left(\frac{2}{5}\pi\right)\right) = 0$ since $\tan\left(\frac{1}{5}\pi\right) = -\tan\left(\frac{4}{5}\pi\right)$ and $\tan\left(\frac{2}{5}\pi\right) = -\tan\left(\frac{3}{5}\pi\right)$ .	1	<b>M1</b>	
	$\Rightarrow x^2 - 10x + 5 = 0$ has roots $\tan^2\left(\frac{1}{5}\pi\right)$ and $\tan^2\left(\frac{2}{5}\pi\right)$ . (AG)	1	<b>A1</b>	
		<b>4</b>		
	$\sec^2 \alpha = 1 + \tan^2 \alpha$	1	<b>M1</b>	
	$y = 1 + x \Rightarrow x = y - 1 \Rightarrow (y - 1)^2 - 10(y - 1) + 5 = 0$	1	<b>M1</b>	
	$\Rightarrow y^2 - 12y + 16 = 0$	1	<b>A1</b>	
		<b>3</b>		

Question	Answer	Marks	Partial Marks	Guidance
11 E	E.g. $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 0 \\ -1 & 0 & 4 \end{vmatrix} = \begin{pmatrix} 8 \\ 4 \\ 2 \end{pmatrix} \sim \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$	2	M1A1	
	$\frac{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}}{\sqrt{4^2 + 2^2 + 1^2}} = \frac{4}{\sqrt{21}}$ (AG)	2	M1A1	
		4		
	$\mathbf{p} = \frac{3}{\sqrt{21}} \left( \frac{4\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{21}} \right) = \frac{1}{7}(4\mathbf{i} + 2\mathbf{j} + \mathbf{k})$	1	B1	
	Line AP: $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$	2	M1A1	
	For Q $1 - 3t = 0 \Rightarrow t = \frac{1}{3} \Rightarrow \mathbf{q} = \frac{1}{3} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$	2	M1A1	
		5		

Question	Answer	Marks	Partial Marks	Guidance
	E.g. $\vec{AB} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ , $\vec{BQ} = \frac{1}{3} \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix}$	1	<b>B1</b>	
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 0 \\ 0 & -4 & 1 \end{vmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$	2	<b>M1A1</b>	
	$\cos^{-1} \frac{\begin{vmatrix} \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \\ \sqrt{21} \cdot \sqrt{21} \end{vmatrix}}{21} = \cos^{-1} \frac{8+2+4}{21} = \cos^{-1} \frac{14}{21} = \cos^{-1} \frac{2}{3}$ (AG)	2	<b>M1A1</b>	
		<b>5</b>		

Question	Answer	Marks	Partial Marks	Guidance
<b>110</b>	Closed curve through pole with correct orientation. Completely correct.	2	<b>B1</b> <b>B1</b>	
	$2 \times \frac{1}{2} a^2 \int_{\frac{1}{2}\pi}^{\pi} (1 - 2 \cos \theta + \cos^2 \theta) d\theta$ $= a^2 \int_{\frac{1}{2}\pi}^{\pi} \left( \frac{3}{2} - 2 \cos \theta + \frac{1}{2} \cos 2\theta \right) d\theta$	2	<b>M1M1</b>	
	$= a^2 \left[ \frac{3}{2} \theta - 2 \sin \theta + \frac{1}{4} \sin 2\theta \right]_{\frac{1}{2}\pi}^{\pi}$	2	<b>M1A1</b>	
	$= a^2 \left( \frac{3}{4} \pi + 2 \right)$	1	<b>A1</b>	
		5		
	$\left( \frac{ds}{d\theta} \right)^2 = a^2 (1 - 2 \cos \theta + \cos^2 \theta + \sin^2 \theta)$	1	<b>B1</b>	
	$= 2a^2 (1 - \cos \theta) = 2a^2 \cdot 2 \sin^2 \frac{1}{2} \theta = 4a^2 \sin^2 \frac{1}{2} \theta \quad (\text{AG})$	2	<b>M1A1</b>	
	$s = 2 \times \int_{\frac{1}{2}\pi}^{\pi} 2a \sin \frac{1}{2} \theta d\theta$	1	<b>M1</b>	
	$= 4a \left[ -2 \cos \frac{1}{2} \theta \right]_{\frac{1}{2}\pi}^{\pi}$	1	<b>A1</b>	
	$= 4\sqrt{2}a$	2	<b>M1A1</b>	
	7			