FURTHER MATHEMATICS

Paper 9795/01
Further Pure Mathematics

Key message

In order to do well in this paper, candidates need to have studied the entire content of the syllabus and ensure that explanation questions are answered fully and carefully.

General comments

There were many good scripts for this paper, with a sizeable minority of the candidates scoring at least 90 of the 120 marks available on the paper. Almost all candidates seemed to have the time to undertake all questions within their scope. In general, candidates scored very heavily on **Questions 1 – 5**, slightly less well on **Questions 7 – 10**, and with marks then tailing off significantly on the final three questions. Only **Question 6**, on Groups, proved to be especially difficult, with many candidates not quite grasping what was required of them. Part of the reason for this would seem to be a reluctance on the part of many candidates to "explain", or justify, results carefully, and such a shortcoming arose in several other places on the paper also.

Overall, nearly all candidates were able to attempt most of the paper, and some presented outstanding scripts, demonstrating an exceptional grasp of the work covered.

Comments on specific questions

Question 1

This was a fairly gentle starter, and was very well completed by most candidates. Almost everyone took the obvious route of multiplying up to determine $\mathbf{A}^3$ and then equating it to the identity matrix $\mathbf{I}$. However, few thought to check that their value of k did indeed give all the correct elements of $\mathbf{I}$. Fortunately, this did not matter in the majority of cases, as they had indeed found $\mathbf{A}^3$ correctly. A few grasped the subtlety of the question's wording, in which they were "given that … $\mathbf{A}^3 = \mathbf{I}$ … ", and hence they actually only needed to find one suitable element of $\mathbf{A}^3$ in order to determine the value of k.

Answers: $k = -7$, det $\mathbf{A} = 1$.

Question 2

This was another straightforward question, and part **(i)** was almost invariably successfully completed. Most candidates immediately recognised that a sum of squares being negative meant that *some* of the roots were complex. However, many failed to notice the second mark available for part **(ii)** and that a further, or more detailed, observation was therefore required.

Answer: **(ii)** One real and two complex (conjugate) roots.

Question 3

This question was very well done. Errors seen were mostly due to arithmetic slips.

Answer: **(i)** $\dfrac{8r}{\left(4r^2 - 1\right)^2}$.

Question 4

This question also proved to be very well-received, with almost all candidates at least knowing what they should be attempting to do. However, there were a few slip-ups to be found. To begin with, several tanh graphs failed to have discernible asymptotes and, in a few cases, they appeared on the sketch, but without any indication of their equations ($y = \pm 1$). A certain degree of generosity was also required on the "explanations" that accompanied the graphs, as only the most diligent took the trouble to equate the two equations of the graphs and re-arrange it to **show** that this was equivalent to $f(x) = 0$. Quite a few candidates lost the mark in part **(ii)(a)** for failing to offer any explanation at all as to why the difference in signs of f(1) and f(1.5) meant that the root α was indeed such that $1 < \alpha < 1.5$. The final 4 marks for part **(ii)(b)** were mostly easily acquired thereafter.

Answer: **(ii) (b)** 1.21876.

Question 5

This was a straightforward differential equations question, and most candidates found it so. There were only a small number of errors, mostly revolving around taking $y = ax^2$ as the trial particular solution, rather than a "full" quadratic.

Answer: $y = A\cos x + B\sin x + 8x^2 - 16$.

Question 6

It would seem that a topic such as group theory is not viewed in the same way as other, more technical and manipulative areas of the course, and some candidates were unsure of what to do with groups. Many candidates tried to prove closure by finding $\begin{pmatrix} p & p \\ p & p \end{pmatrix}^2$, which is missing the point somewhat. Also, matrix multiplication is known to be associative, so showing it the long way is unnecessary. After that, quite a few candidates assumed that the identity of this group had to be $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and that the inverse of $\begin{pmatrix} p & p \\ p & p \end{pmatrix}$ was simply going to be $\begin{pmatrix} p & p \\ p & p \end{pmatrix}^{-1}$. Methods for finding subgroups of order 2 and 3 were even less certain, and there were several appeals to Lagrange's Theorem which, of course, only applies to finite groups.

Answer: **(ii)** Subgroup is $\{\boldsymbol{E}, \boldsymbol{A}\}$ where $\boldsymbol{E} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ and $\boldsymbol{A} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$.

Question 7

This was the last of the straightforward questions and proved to be a popular question. The only major obstacle was in the interpretation of the question's wording, where the phrase "all significant features" appears. This is a verbatim quote from the syllabus, where it is made clear that "all significant features" include any turning points of the curve at hand. A small but significant number of candidates lost the four marks for determining the turning points by making no attempt to find them. The only other point to note is that the equation of the oblique asymptote was often incorrectly found, due to algebraic or numerical slips, and this frequently led to difficulties on the sketch as to the behaviour of the curve relative to this asymptote as $x \to -\infty$.

Question 8

Candidates scored well on this question also, and generally lost marks only due to small algebraic slips. For future reference, it is extremely helpful, both to the Examiner and the candidates themselves, when candidates **say** what they are doing when adding or subtracting (multiples of) one row from another.

Answers: $k = 3$ (inconsistent); $k = 5$ (consistent).

Question 9

This vectors question was found to be reasonably comfortable with most of the work correct in principle and subject only to minor slips (in calculating vector products, for instance). The most frequent mistake was an incorrect statement of the formula for the volume of a tetrahedron.

Answers: **(a)** $335\frac{1}{6}$; **(b) (i)** $\mathbf{r} \bullet \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} = 1$; **(ii)** $\mathbf{r} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$.

Question 10

This induction question was well-handled by most candidates to begin with, but the slightly different nature to the usual type of induction proof differentiated well between those who could adapt to its needs and those who could not. The first two parts were usually done successfully, being fairly routine in nature, and most candidates spotted that the required conjecture was simply that $u_n = \cos n\theta$. (The idea comes from the Chebyshev polynomials.) At this point, candidates need to assume both u_n and u_{n-1} before confirming the validity of the conjecture for u_{n+1} and this is indicative of a form of induction called strong induction. Candidates were not penalised twice for assuming only u_n (they could still earn the final mark for the “induction round-up” without this extra assumption), but they should have spotted that the recurrence definition required the previous two terms, and they were penalised at this early stage for not noticing this. To be fair, around half of all candidates *did* spot the need for two prior terms. Other candidates who lost the final mark generally did so by failing to offer anything more than a concluding remark along the lines of “proof follows by induction”; there will always be a mark for an explanation of the inductive logic, and those candidates who simply follow a formulaic approach to a proof by induction will need to be a little more careful in their demonstration of their understanding of the process.

Answers: **(ii) (a)** $u_2 = 2\cos^2\theta - 1$; $u_3 = 4\cos^3\theta - 3\cos\theta$; **(b)** $u_n = \cos(n\theta)$.

Question 11

On this paper the final few questions are intended to be more demanding, testing both candidates’ genuine understanding and their technical and manipulative skills. The reduction formula of part **(i)** required candidates to decide how to split the $\sec^n t$ term in order to integrate by parts. The only viable possibility is as $\sec^{n-2}t \times \sec^2 t$, and this part proved so demanding precisely because candidates were given no help from within the question as to how best to proceed. Part **(ii)** was then broken up into two stages and most candidates successfully showed that $S = \pi I_5$. It was a little surprising to see that most candidates then went on to employ the given reduction formula twice (as required) in *descending* order – i.e. finding I_5 in terms of I_3 and then I_3 in terms of I_1 – rather than in *ascending* order. Determining I_1 first and *then* finding I_3 and I_5 almost invariably leads to fewer errors (because of the lack of nested brackets) and, where errors were made, this was almost always due to this reason.

Answer: **(ii)** $\frac{1}{144}\left(68 + 27\ln 3\right)\pi$.

Question 12

This was a relatively short question. Part **(i)** was helpfully structured, although candidates again frequently lost the mark awarded for explaining why a^2 was $1+\sqrt{2}$, even when they then went on to explain the choice of positive sign when taking its square-root (for which there was no mark, due to the question stating that a was positive). The similar form for b was, of course, $\sqrt{\sqrt{2}-1}$ and not just $\dfrac{1}{\sqrt{\sqrt{2}+1}}$. In part **(ii)**, the method mark was for considering that z_2 was a complex number in the second-quadrant of the complex plane, and many candidates failed to spot this. Even amongst those with correct args for z_2, the method for finding the least n for which the given result holds was often not at all clear. Those who arrived at the answer 14 were generally given the benefit of the doubt, even if their working failed to demonstrate that this was indeed the *least* such integer.

Answers: **(i)** $\sqrt{\sqrt{2}-1}$; **(ii) (a)** $\frac{7}{8}\pi$; **(b)** Least n is 14.

Question 13

There was a rather nice thread throughout this question, and the less successful attempts generally were those which failed to spot, or exploit, this. Most attempts at the opening identity began with the right-hand-side, noting that $\sec^2 x = 1 + t^2$, and then multiplying throughout the fraction by $\cos^2 x$. Part **(i)(b)** was rather poorly done, possibly because candidates were put off by the unseemly appearance of the given expression. Far too few efforts attempted to find its derivative using the Chain Rule, and this generally lost them all four of these marks. In part **(ii)(a)**, it had been expected that candidates would find r^2 in the form $\dfrac{k}{2-\sin 2x}$ (as per the thread mentioned earlier) but many resisted this form. Whilst it had been anticipated that candidates would use the properties of the sine function, many of them preferred to attempt to use calculus, not always terribly well. Most incomplete final answers missed the value $\theta = \frac{5}{4}\pi$. For a variety of reasons, few got around to making much of an attempt at the final part of the question. Of those who did, many seemed not to notice the very final demand.

Answers: **(i) (b)** $k = -\sqrt{3}$; **(ii) (a)** $(12, \frac{1}{4}\pi)$ and $(12, \frac{5}{4}\pi)$; **(b)** $8\pi\sqrt{3}$, $16\pi\sqrt{3}$.

FURTHER MATHEMATICS

Paper 9795/02
Further Application of Mathematics

Key Message

Candidates should be reminded to conform to the rubric on the front of the paper, giving their answers to 3 significant figures, unless otherwise specified, and using a value of 10 ms^{-1} for g.

General Comments

The vast majority of candidates responded well to this paper. There were a large number of very good scripts and also some of an outstanding quality. Candidates seemed to have covered most areas of the syllabus and good work was seen on all questions on the paper. There was evidence of good preparation for this year's paper. Some changes in the structure of the paper had been made since last year and these helped candidates. The first three questions in each section were more structured and all candidates found they could work confidently at these questions. The order of the two sections of the paper was reversed; this helped candidates to make a strong start on the paper, although the disparity between performance on probability and mechanics was much less marked this year. The presentation of work showed an improvement with more candidates presenting their work in an orderly and logical fashion, making it easy to follow their reasoning. Arithmetical accuracy and algebraic manipulation were mostly of a good standard. Some candidates would have made life easier for themselves by simplifying expressions before proceeding. Most candidates seemed to have to have adequate time to complete the paper and to check their work.

Comments on specific questions

Question 1

Most candidates were able to gain the majority of marks on this first question, with only a few having the variance as 145, rather than 43. A small number of candidates gave the final answer as the complement of the correct answer.

Answer: **(i)** N(10, 43) **(ii)** 0.936

Question 2

Almost all candidates gained all three marks on the first part, the exception being those who found P(X < 5). There were many correct answers to part **(ii)**. The most common error was not to use a continuity correction. In the final part, when previous work was correct, the expected comment was that the result was unreliable because the Poisson mean was not large enough. If, because of a previous error the obtained results were close, marks were awarded for a sensible comment on the results that the candidate had obtained.

Answers: **(i)** 0.0174 **(ii)** 0.0269**.**

Question 3

The vast majority of candidates successfully obtained the unbiased estimate of population variance. The form in which the data was given to them should have enabled them to achieve it with minimal arithmetic. It was, therefore, surprising to find a small number of candidates multiplying and dividing by n_1 and n_2, unnecessarily. Many obtained the 95% confidence interval correctly. The errors that did occur were: an incorrect expression for the standard error of the differences between means, incorrect number of degrees of freedom and/or critical value. A small number of candidates thought that they were dealing with a normal distribution. The marks for comments could be earned on a follow through basis if the confidence interval obtained was incorrect. The method mark was for stating whether 0 was in (or out of) their confidence interval. This could be implied by what they said. A second mark was for coming to an appropriate conclusion, having earned the method mark.

Answer: **(i)(b)** (–0.252 or –0.251,5.65).

Question 4

Most candidates made a reasonable attempt at the first part of the question. The better candidates were able to cope with the double integration by parts successfully and produce the correct answer after substituting limits. Part **(ii)** required the recognition that E(T) was θ^2 and to find an expression for E(T), in order to form an equation that could be rearranged to find k. Only the best candidates managed to do this accurately.

Answers: **(i)** $2\theta^2$ **(ii)** $k = \dfrac{1}{2n}$.

Question 5

A large number of candidates were able to form and sum an appropriate geometric progression for the moment generating function for the geometric distribution. The method of using it to find the first and second moments, and hence the variance, was well known. It was here that many candidates failed to simplify algebraic expressions and made the second differentiation unnecessarily difficult for themselves. Correct answers, although not in a simple form earned full marks. Most candidates saw, in the final part, that μ was 6 and σ was $\sqrt{30}$, but only the best were able to deduce that P($Y \le 11$) was required and were able to find it correctly.

Answers: **(ii)(a)** $\dfrac{1}{p}$ **(b)** $\dfrac{q}{p^2}$ **(iii)** 0.865.

Question 6

The solutions to the first part were often only partially correct and lacked key details. There were two ways of approaching this part. One approach was to find the cumulative distribution function of X, transform this to the cumulative distribution function of Y, then differentiate to give the required probability density function of Y. The alternative was to write the result $\int_0^{\frac{\pi}{2}} \frac{2}{\pi}\,dx = 1$ and use the given substitution to obtain the integral in terms of y, from which the probability density function of Y could be determined. Both methods were evident in approximately equal proportions. Finding the median of Y proved to be a good source of marks for the majority of candidates. In the final part, most could write down the required integral representation of E(Y), although finding its value caused some problems. The best could write down the result of integrating by inspection. Some others successfully used a substitution, while others quoted incorrect results, which usually included an inverse sine function.

Answers: **(ii)(a)** $\dfrac{1}{\sqrt{2}}$ **(b)** $\dfrac{2}{\pi}$.

Question 7

The majority of candidates could use the principle of conservation of energy to obtain the printed answer for the first part. There were a few instances of candidates using g = 9.8, which resulted in the loss of a mark in this question (but not for continued use in other questions). Weaker candidates only found one of the components of acceleration in part **(ii)**. Where errors occurred in the radial and/or transverse component, correct use of Pythagoras earned a follow through mark.

Answers: **(ii)** Radial 10.1 ms^{-2}, Transverse 6.43 ms^{-2}, Resultant 12.0 ms^{-2}.

Question 8

The majority of candidates were successful on this question. Parts **(i)(a)** and **(ii)** usually produced the correct results. In part **(i)(b)** a sign error was quite common.

Answers: **(i)(a)** $\frac{7\sqrt{3}}{5} = 2.42$ ms^{-1} **(b)** $\frac{147\sqrt{3}}{100} = 2.55$ Ns **(ii)** 4.26 ms^{-1} at 34.7° to wall.

Question 9

Most candidates made sensible attempts to write the required vectors for parts **(i)** and **(ii)**. There were some sign errors and, occasionally, sine and cosine were interchanged. In part **(ii)** the –13**j** was sometimes omitted. Alert candidates moved rapidly from part **(i)** to part **(ii)** by multiplying by *t* and subtracting 13**j.** In part **(iii)** some attempt had to be made to describe the relative situation by saying, for example, *B* is considered to be stationary and *A* is moving relative to *B*, so that when the two are closest, the relative velocity vector is perpendicular to the relative position vector. Whether or not this was stated, most candidates were able to make a good effort at finding the required equation, using the scalar product and equating to zero, which they then solved to find the time of closest approach.

Answers: **(i)** (10 sin42° – 15)**i** + (10 cos42°)**j** or –8.31**i** + 7.43**j.**
(ii) (10 sin42° – 15)*t***i** + (10 cos42°*t* – 13)**j** or –8.31*t***i** + (7.43*t* – 13)**j.** **(iv)** 1247.

Question 10

Most candidates were able to use Hooke's law and equate tensions in order to find the length *AP*, usually correctly. There were many good answers to the second part. Weaker candidates did not manage this part, mainly because they did not consider the general position. The final part produced many correct answers, mostly from those who used the equation $y = a\cos\omega t$. There were some sign errors and those who worked from the centre, rather than the end, produced rather muddled work.

Answers: **(i)** 3.5 m. **(iii)** 0.331 seconds.

Question 11

There were a pleasing number of correct solutions to this projectile question. In the first part, a number of candidates confused results where the projectile landed on a horizontal plane with this case, where it landed on an inclined plane. Able candidates, who quickly found the time of flight by considering a *y*-axis perpendicular to the inclined plane, then switched to a horizontal *x*-axis, to find the horizontal displacement, before invoking $R = \frac{x}{\cos\alpha}$ to obtain the range on the inclined plane. This neatly gave the result, using the appropriate compound angle formula. Many were able to write the correct result for part **(ii)**, without undue calculation, as the single mark indicated should be the case. There were good attempts at the final part, even when things had broken down in the first part of the question.

Answers: **(ii)** $\frac{2V^2\sin\theta}{(g\cos^2\alpha)(\cos\theta\ \ \cos\alpha + \sin\theta\sin\ \alpha)}$ **(ii)** 50.2°.

Question 12

The first part of this question was done successfully by the vast majority of candidates. The usual errors were the omission of g, when finding the component of weight acting down the track, or the omission of this term altogether. In the second part some candidates tried to use constant acceleration formulae, while others wrote their acceleration as $\frac{\mathrm{d}x}{\mathrm{d}t}$. The rare candidate got away with the latter by finding the time taken to reach a speed of $\frac{3}{4}V$ before going on to find the distance. A pleasing number wrote the equation of motion with $v\frac{\mathrm{d}v}{\mathrm{d}x}$ for the acceleration. Only the best were able to write the equation in a form which they could integrate and obtain the correct answer.

Answers: **(i)** 16 ms^{-1} **(ii)** 846 m.