FURTHER MATHEMATICS

Paper 9795/01
Further Pure Mathematics

General Comments

There was a significant increase in the overall quality of the candidates' responses from last year, with almost a quarter of all candidates scoring at least 100 or more of the 120 marks available on the paper. Although the paper was both long and difficult, almost all candidates seemed to have sufficient time to undertake all questions within their scope. Candidates generally scored most of the marks on **Questions 1** and **2**; thereafter, apart from **Questions 6** and **13**, candidates scored well on each of the remaining questions, gaining on average around two-thirds to three-quarters of the marks available on **Questions 3** – **10**, then dropping slightly on the longer and technically more demanding **Questions 11** and **12**.

Question 6, on the topic of '*Groups*' again this year, was found difficult, testing as it did candidates' abilities to construct their own reasoned arguments, and the short **Question 13** required a more considered induction hypothesis than is usually the case.

Overall, nearly all candidates found plenty that they could do, and many presented outstanding scripts, demonstrating an exceptional grasp of the work covered. There were several particular instances of ingenuity and skill to be found amongst the scripts.

Comments on specific questions

Question 1

This was a very routine starter, and was very successfully completed by almost all candidates. The usual pitfall of writing $\sum_{r=1}^{n} 1 = 1$, rather than n, was generally avoided.

Question 2

This was another straightforward question, and most candidates coped perfectly well with its demands. A few even circumnavigated the need to use a double-angle formula by integrating $2\sin\theta\cos\theta$ directly as $\sin^2\theta$. Only a few mistakes were made; some quoted a completely incorrect area integral formula, and occasionally the $\frac{1}{2}$ was either forgotten at the outset or disappeared along the way.

Answer: $\frac{1}{4}\pi + \frac{1}{2}$.

Question 3

Apart from a few candidates who forgot to express $\frac{dy}{dx}$ "in terms of y only", part **(i)** was capably handled. Part **(ii)**, however, received a bit more of a mixed response. The intention was that candidates would spot that the result simply involved reversing the result of part **(i)** and changing a label or two – indeed, many spotted this straight away. The remaining candidates attempted a variety of substitutions; the more thoughtful ones of which (such as $t^2 = \sinh\theta$) were almost invariably successful.

Answers: **(i)** $\frac{\sqrt{1+y^4}}{2y}$; **(ii)** $\sinh^{-1}(t^2) + C$

Question 4

A lot of marks were scored on this question, though not necessarily by all candidates. Many did not seem to know what "suitable quadratic" was required in part **(i)**, even though the method sought for here is clearly mentioned in the syllabus. As a result, these candidates' part **(ii)** generally consisted of a calculus-based approach. However, it was decided to allow this as an alternative approach.

Answers: **(i)** $-\frac{1}{6} \le y \le \frac{1}{2}$; **(ii)** $(1, \frac{1}{2})$ and $(-3, -\frac{1}{6})$.

Question 5

This question was done well by almost all candidates. Expected to "write down" the two matrices in part **(i)**, a small number of candidates wasted time trying to construct them from scratch, and a few of these unfortunately got these standard results wrong. Those who did manage this small introductory hurdle usually went on to score 5 or 6 of the marks available in part **(ii)** – the only mark commonly lost was due to writing the two reflection matrices down in the incorrect (left-to-right, rather than right-to-left) order. The trigonometric addition formulae were handled very comfortably overall.

Answers: **(i) (a)** $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$; **(b)** $\begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix}$; **(ii)** a rotation (about O) through $(\phi - \theta)$

Question 6

As mentioned in the introduction, this was one of the two questions found difficult by many candidates. The question was essentially a mix of the use of quotable results and the requirement to formulate a logical argument for oneself. Part **(i)** involved the former, with "Lagrange's Theorem" or an equivalent statement of what it entailed being appropriately offered by the majority. Just a few, however, failed to write down *all* the factors of 12 – usually an omission of the 1 or the 12, or both. In part **(ii)**, many candidates produced an admirable, logical response to the proof of the required result, even though it would have been helpful if more candidates had offered some indication as to which of the three given conditions they were using at each stage of their argument. Just a few marks were lost by candidates who started with the answer and then worked backwards towards one of the given conditions, often assuming the other ones along the way – this is actually quite a serious error (and a good point for classroom discussion).

The final part could be dealt with by simply quoting the result "not abelian ⇒ not cyclic" and then showing that the group isn't abelian; this was the expected approach, but few candidates seemed to consider it. Successful candidates generally set up a contradiction that amounted to the same thing; the unsuccessful ones usually tried to produce all 12 elements of the group and show that none acted as a generator, but this proved too lengthy a task and most efforts were unsuccessful.

Answers: **(i)** 1, 2, 3, 4, 6 or 12.

Question 7

Part **(i)** of this question was clearly standard fare for almost all candidates, and most gained all five of the marks here. Part **(ii)**, however, proved to be surprisingly tricky. The only mark widely gained was the one for finding the numerical value of $\tan 4\theta$. Thereafter, most candidates left a blank space, resorted to their calculator, or invented spurious arctan results; given how well, and how naturally, most candidates had used the addition formulae for sine and cosine in **Question 5**, it was surprising to find that the corresponding one for tangent appeared so seldom.

Question 8

This was another popular and high-scoring question, although numerical slips and sign errors were relatively common, and just a few candidates thought that the derivative of $3e^{x-1} + 1$ was $3(x-1)e^{x-2}$.

Answers: **(i)** 5; **(ii)** –12 ; **(iii)** $2 + 3(x-1) + \frac{5}{2}(x-1)^2 - 2(x-1)^3$; **(iv)** 2.323

Question 9

On the whole, this question was successfully undertaken by the majority of candidates, although completely correct solutions were quite rare. Part **(i)** was straightforward, although many candidates made heavy work of establishing the given result. Part **(ii)** benefitted from having two standard solution approaches – the "integrating-factor" approach and the less commonly used "1st-order linear with constant coefficients" method. The biggest hindrance to a completely successful outcome was in the way (or the order) in which the constant of integration was dealt with: it sometimes failed to appear at all, sometimes *after* the initial condition had been used, and occasionally was simply ignored when it came to multiplying throughout an equation by, say, e^{3x}.

Answer: **(ii)** $y^3 = \dfrac{1}{3x + 1 + 7e^{3x}}$.

Question 10

This vectors question was well-handled by most candidates, at least in principle – valid methods were almost always employed, but slip-ups in calculating scalar or vector products were rather commonplace.

Answers: **(i)** 26 ; **(ii)** $Q = (16, -16, -34)$, 21 ; **(iii)** $10x - 9y + z = 39$

Question 11

Most candidates scored well on some parts of this question, but tended to lose marks "in the detail". In part **(i)**, many candidates effectively just asserted that the required argument was $\frac{5}{12}\pi$, rather than justify that it was so – since this result was essentially given to them in the question, this was penalised. More marks were lost in part **(ii)(b)**, as many misinterpreted the question to say, "Determine z_1, z_2, z_3 in the form $a + ib$." Relatively few spotted immediately that the answer was just w.

Part **(c)** also saw its fair share of lost marks, through insufficient care being taken in justifying that the triangle of roots was equilateral. Part **(d)**, however, was intended to require a bit of clear-headed thought, and few candidates spotted what was required – most offerings clearly had magnitudes greater than 1, which a bit of thought would have quickly shown to be unsuitable.

Answers: **(i)** $|w| = \sqrt{2}$, arg $w = \frac{5}{12}\pi$; **(ii) (a)** $\left(\sqrt{2}, \frac{5}{36}\pi\right)$, $\left(\sqrt{2}, \frac{29}{36}\pi\right)$, $\left(\sqrt{2}, -\frac{19}{36}\pi\right)$;
(b) $\left(\sqrt{3} - 1\right) + i\left(\sqrt{3} + 1\right)$ or "w" ; **(c)** $\sqrt{6}$; **(d)** $\exp\left(-\frac{5}{36}\pi\right)i$

Question 12

Part **(i)** of this question was the most technically demanding on the whole paper, requiring of candidates that they spot how to split up the integrand to begin with and then to be able to integrate the second "part", $x\sqrt{16 + x^2}$, correctly. Many who struggled with this part showed excellent exam-technique by being completely undeterred from continuing with their efforts to the remaining parts of the question. In **(ii)(a)**, many candidates' sketch of the given spiral went beyond the given domain; in the remainder of the question, the better performing candidates were those who realised that they could use the given reduction formula from part **(i)**.

Answer: **(ii) (b)** $\frac{1423}{60}$.

Question 13

The overwhelming majority of candidates found this question very difficult and made little progress. However, it should be noted that the part of the syllabus that deals with induction does indicate that candidates may be required to do a little experimentation before conjecturing, and then proving, an induction result, and it was the unwillingness to experiment a little that was the downfall of most candidates. The other major hurdle to get over was that the induction hypothesis “assume the result is true for $n = k$”, rattled off by rote, was of little value – a far more carefully considered assumption regarding the form of these numbers was required; it was the fact that they all begin and end in the same way that was essential to incorporate into the hypothesis. Quite a few candidates spotted what was going on but were unable to formulate it into an inductive proof, most ending up trying to establish the result *constructively*. A further hurdle to successful progress, if one were needed by this point, was the inability to be able to say how R_{k+1} was to be gained from R_k – there are, in fact, two ways to do this. This question *was* intended as a top-end discriminator and, in fact, a few candidates scored 5 or 6 marks on it, affording them the opportunity to demonstrate real star-quality.

FURTHER MATHEMATICS

Paper 9795/02
Further Applications of Mathematics

General Comments

Candidates responded well to this paper. There were a large number of very good scripts and also some of an outstanding quality. The disparity between performance on probability and mechanics was less marked this year, indeed a rather higher number than previously performed better on the mechanics section than on the probability section. The presentation of work was of a good standard, with candidates, generally, presenting their work in an orderly and logical fashion, making it easy to follow their reasoning. Arithmetical accuracy was mostly good. Most candidates gave numerical answers to at least three significant figures, with only the occasional exception which was penalised in this respect. Using g as 9.8 was not penalised on the rare occasion when it occurred. Algebraic manipulation was mostly of a good standard. The tendency of candidates to persevere with unsimplified algebraic expressions, unfortunately, still persisted. This, in some cases, prevented further progress in the more intricate and demanding questions.

Most candidates seemed to have adequate time to complete the paper and to check their work. A few candidates could have improved their total mark by checking work more rigorously, for example when a sine or cosine value was greater than 1, possibly due to a wrong sign in their equation.

Candidates seemed to have covered most areas of the syllabus and good work was seen on all questions on the paper. There was evidence of good preparation for this year's paper.

Comments on specific questions

Section A: Probability

Question 1

Most candidates did well on this question and so got off to a good start on the paper. Part **(i)** was more problematic for the less confident candidates than part **(ii)**. In part **(ii)** a small number of candidates used unnecessarily laborious methods for differentiating, involving the quotient rule.

Answers: **(ii)** $\frac{1}{k}$ $\frac{1}{k^2}$.

Question 2

Good marks were scored by many candidates on this question. In part **(i),** a number of candidates assumed that $a+3b=1$ and showed that $\mathrm{E}(a\overline{X}+b\overline{Y})=\mu$, rather than showing the converse. On this occasion it was not penalised. In part **(iii)** the better candidates produced solutions involving completing the square, or properties of quadratic functions, which was good to see.

Answer: **(iii)** $\frac{4\sigma^2}{13n}$.

Question 3

There were plenty of completely correct confidence intervals for both part **(i)** and part **(ii)** from those who remembered to use z when the variance is known and t when it is unknown. There were a significant number of candidates who either used z in both parts, or who used t in both parts. In either case they could not make a sensible comment for part **(iii)**, as they had identical confidence intervals. Part **(iii)** required a sensible comment referring to the fact that 1.8 was outside the first confidence interval, but inside the second. The mark for part **(iii)** could, however, be earned if an earlier arithmetical slip resulted in a different conclusion.

Answers: **(i)** (1.57, 1.78) **(ii)** (1.51, 1.84).

Question 4

The bookwork in parts **(i)** and **(ii)** was well done by the many candidates who had thoroughly learned the material. Only the very best candidates were able to cope with the conditional probability in part **(iii)**. The vast majority seemed to be unaware that $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$. Most seemed to find P($B$) only for 1 mark and some managed $\mathrm{P}(A \cap B)$ for 2 marks, but few knew that they needed to form the quotient. The occasional solution from candidates who managed to find the exact answer was most impressive.

Answer: **(iii)** 0.848.

Question 5

In part **(i)**, the appropriate normal approximation was the normal approximation to the binomial distribution. Those who used the Poisson distribution gained 3 of the first 6 marks only. Marks were lost, in both part **(a)** and part **(b)**, by those who either used no continuity correction, or who used the incorrect continuity correction. Part **(ii)** required the normal approximation to the Poisson distribution, and also a continuity correction. Those not using the latter frequently obtained the correct answer, but forfeited 1 mark for incorrect working.

Answers: **(i) (a)** 0.915 **(b)** 0.0304 **(ii)** 46.

Question 6

In part **(i)** some good sketch graphs were drawn of the cubic function, showing tangency at the origin. However, at this level, it was surprising to find many poorly drawn sketches. Most looked parabolic and the occasional offering was triangular. By contrast, part **(ii)** was done much better. The mean was usually found correctly and only a few candidates confused the mode with the median. The mark for part **(iii)** was generously awarded to those who stated that the mean being less than the mode implied negative skewness, even if their sketch did not exhibit this property. There was some confusion, in the minds of a significant minority, between what constituted positive and negative skewness. Only the best candidates were able to successfully negotiate part **(iv)**, since most could not determine the correct limits for their integration. These limits were required to earn the initial method mark for this part of the question.

Answers: **(ii)** 1.8 , 2 **;** **(iv)** 0.64**.**

Section B: Mechanics

Question 7

Most candidates were able to find the value of k correctly in part **(i)** and derive the appropriate differential equation in part **(ii)**. Many candidates were able to separate the variables and thus solve the differential equation correctly in part **(iii)**, although some had to use partial fractions in order to do so, rather than observing where the numerator was the derivative of the denominator, in one of their integrals.

Answer: **(iii)** $60\ln\left(\dfrac{91}{51}\right)$ or 34.7 seconds.

Question 8

There were successful solutions of this relative velocity problem, either by using cosine and sine rules on distance and velocity triangles, or by equating easterly and northerly displacements of the two vessels at time *t* hours. In some solutions, using either method, candidates had the destroyer finishing 3 km west of the aircraft carrier, rather than 3 km east of the aircraft carrier, as stated in the question. Consequently all accuracy was lost and only method marks were gained. The method whereby two simultaneous equations were solved produced a rather awkward trigonometric equation to solve, whereas the approach using cosine and sine rules generally had a higher success rate. Three figure bearing notation was not required for the final mark.

Answer: 023°.

Question 9

There were a good number of correct answers to this question. The layout of solutions was found difficult for quite a few candidates. Their solutions often began by writing $T\cos\theta = mg$ and $T\sin\theta = mr\omega^2$ immediately, clearly rehearsing what they had learned from their notes on this topic. Part **(i)** only required the first of these, from which it was clear that θ could never be 90°, hence $0 < \cos\theta < 1$ and thus $T > mg$. Other arguments were acceptable, provided that it was stated that $t\cos\theta = mg$ somewhere in the argument. Part **(ii)** required the second of the standard equations. A few candidates made life difficult for themselves by working with the speed *v* rather than the angular speed ω, which was mentioned in the question, or by writing $\sin\theta$ in terms of *r* and *l*. The result required in part **(iii)** was usually produced, although sometimes it had appeared in working in part **(ii)**. Since these were B marks, they were awarded wherever they were earned. Those few who did not manage to obtain the result were, nevertheless, able to gain the final mark by substituting the appropriate values.

Answers: **(ii)** $ml\omega^2$ **(iii)** 4.47 rad/s.

Question 10

It was hoped that, by making *P* and *Q* move perpendicularly after the collision, most candidates would be able to write the simple conservation of linear momentum equation. A considerable proportion of the candidates did not capitalise on this, having *Q* moving at an angle, other than 90°, to the line of centres. This made progress on part **(i)** impossible and often meant low scoring on part **(ii)** as well. Those with a correct first part usually managed to produce correct work in part **(ii)**, although some arithmetical mistakes occurred in the energy equation.

Answers: **(i)** $\frac{1}{3}$ **(ii)** 60°.

Question 11

In this question, candidates were able to make a good attempt at applying Hooke's law to calculate the tension in the elastic strings. They were then able to form a Newton II equation and, with a little elementary trigonometry, produce the given result for part **(i)**. One error in this first part, was to think that the particle was hanging from the strings, below the level of *A* and *B*. This emphasises the need to read questions carefully. Those who did not manage to derive the result in the first part were able to use it, however, to do the next two parts of the question. Most candidates were successful in parts **(ii)** and **(iii)**.

Answers: **(ii)** $2\pi\sqrt{\frac{a}{8g}}$ seconds.

Question 12

There was much good work on this final question. The majority of candidates were able to derive the equation of the trajectory of a projectile, from first principles. In part **(ii),** many realised that the equation of the trajectory was a quadratic equation in $\tan\alpha$, but did not write it in standard quadratic form. Consequently, when trying to use the discriminant to find the condition that $\tan\alpha$ was real, they made one, or more, algebraic or arithmetical errors. The omission of the *y* term was a common error, which meant that the final inequality could not be deduced. The printed result, however, could be used to make progress on the final part. The usual approach was to solve the trajectory equation and the bounding parabola equation simultaneously, thus producing a quadratic equation in *x*, which most were able to solve. Only the best candidates, however, were able to use $x = R\sec\beta$ and perform the required manipulation, in order to find the range, *R*, up or down the inclined plane.