UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS

Pre-U Certificate

MARK SCHEME for the May/June 2012 question paper

for the guidance of teachers

9795 FURTHER MATHEMATICS

9795/01 Paper 1 (Further Pure Mathematics), maximum raw mark 120

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

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1		$\sum_{r=1}^{n}\left(r^2-r+1\right)=\sum_{r=1}^{n}r^2-\sum_{r=1}^{n}r+\sum_{r=1}^{n}1$ Splitting summation and use of given results	M1	
		$=\frac{1}{6}n(n+1)(2n+1)-\frac{1}{2}n(n+1)+n$ 1st for Σr^2; 2nd for Σr & $\Sigma 1 = n$	B1 B1	
		$=\frac{1}{3}n(n^2+2)$ legitimately	A1	[4]
2		$A = k\int(\sin\theta+\cos\theta)^2\,\mathrm{d}\theta$ including squaring attempt; ignore limits and $k \neq \frac{1}{2}$	M1	
		$=\frac{1}{2}\int(1+\sin 2\theta)\,\mathrm{d}\theta$ for use of the double-angle formula **OR** integration of $\sin\theta\cos\theta$ as $k\sin^2\theta$ or $k\cos^2\theta$	B1	
		$=\frac{1}{2}\left[\theta-\frac{1}{2}\cos 2\theta\right]_0^{\pi/2}$ **ft** (constants only) in the integration; MUST be 2 separate terms	A1	
		$=\frac{1}{4}\pi+\frac{1}{2}$	A1	[4]
3	(i)	$y=(\sinh x)^{\frac{1}{2}} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}(\sinh x)^{-\frac{1}{2}}.\cosh x$ **OR** $y^2=\sinh x \Rightarrow 2y\frac{\mathrm{d}y}{\mathrm{d}x}=\cosh x$	M1 A1	
		$=\frac{\sqrt{1+y^4}}{2y}$	A1	[3]
	(ii)	$\int\frac{2y}{\sqrt{1+y^4}}\,\mathrm{d}y=\int 1.\mathrm{d}x$ By Sep^{g}. Vars. in **(i)**'s answer	M1	
		$\Rightarrow x=\int\frac{2y}{\sqrt{1+y^4}}\,\mathrm{d}y$	A1	
		But $x=\sinh^{-1}y^2$ so $\int\frac{2t}{\sqrt{1+t^4}}\,\mathrm{d}x=\sinh^{-1}(t^2)+C$ condone missing " $+C$ "	A1	
		ALT.1 Set $t^2=\sinh\theta$, $2t\,\mathrm{d}t=\cosh\theta\,\mathrm{d}\theta$ **M1** Full substn. $\int\frac{2t}{\sqrt{1+t^4}}\,\mathrm{d}t=\int\frac{\cosh\theta}{\sqrt{1+\sinh^2\theta}}\,\mathrm{d}\theta$ **A1** $=\int 1.\mathrm{d}\theta=\theta=\sinh^{-1}(t^2)$ **A1** **ALT.2** Set $t^2=\tan\theta$, $2t\,\mathrm{d}t=\sec^2\theta\,\mathrm{d}\theta$ **M1** Full substn. $\int\frac{2t}{\sqrt{1+t^4}}\,\mathrm{d}t=\int\frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}}\,\mathrm{d}\theta$ **A1** $=\int\sec\theta\,.\mathrm{d}\theta$ $=\ln\lvert\sec\theta+\tan\theta\rvert = \ln\lvert t^2+\sqrt{1+t^4}\rvert$ **A1**		[3]

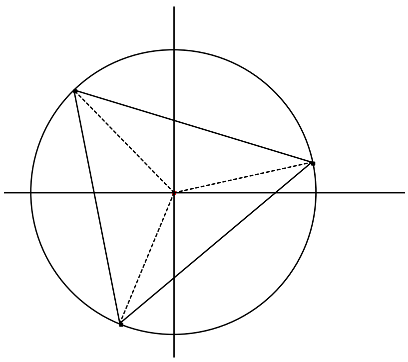
4	(i)	$y = \dfrac{x+1}{x^2+3} \Rightarrow y.x^2 - x + (3y-1) = 0$ Creating a quadratic in x	M1	
		For real x, $1 - 4y(3y - 1) \geq 0$ Considering the discriminant	M1	
		$12y^2 - 4y - 1 \leq 0$ Creating a quadratic **inequality**	M1	
		For real x, $(6y + 1)(2y - 1) \leq 0$ Factorising/solving a 3-term quadratic	M1	
		$-\frac{1}{6} \leq y \leq \frac{1}{2}$ **cso**	A1	
		NB lack of inequality earlier with unjustified correct answer loses only the 3rd **M** mark		[5]
	(ii)	$y = \frac{1}{2}$ substd. back $\Rightarrow \frac{1}{2}\ (x^2 - 2x + 1) = 0 \Rightarrow x = 1$ $[\ y = \frac{1}{2}]$	M1 A1	
		$y\ -\frac{1}{6} =$ substd. back $\Rightarrow -\frac{1}{6}\ (x^2 + 6x + 9) = 0 \Rightarrow x = -3$ $[y = -\frac{1}{6}]$	M1 A1	
		Allow alternative approach via calculus: $\dfrac{dy}{dx} = \dfrac{-x^2 - 2x + 3}{(x^2+3)^2}$ **M1** Solving quadratic to find 2 values of x **M1** Then **A1 A1** each pair of correct (x, y) coordinates		[4]
5	(i)	**(a)** $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$	B1	
		(b) $\begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix}$	B1	[2]
	(ii)	$\begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ Multn. of 2 reflection matrices. Correct order.	M1 M1	
		$= \begin{pmatrix} \cos\phi\cos\theta + \sin\phi\sin\theta & \cos\phi\sin\theta - \sin\phi\cos\theta \\ \sin\phi\cos\theta - \cos\phi\sin\theta & \cos\phi\cos\theta + \sin\phi\sin\theta \end{pmatrix}$		
		$= \begin{pmatrix} \cos(\phi-\theta) & -\sin(\phi-\theta) \\ \sin(\phi-\theta) & \cos(\phi-\theta) \end{pmatrix}$ Use of the addition formulae; correctly done	M1 A1	
		… giving a Rotation (about O) through $(\phi - \theta)$ acw [or $(\theta - \phi)$ cw]	M1 A1	
		Those who get the initial matrices in the wrong order, can get 5/6, losing only that **M** mark		[6]

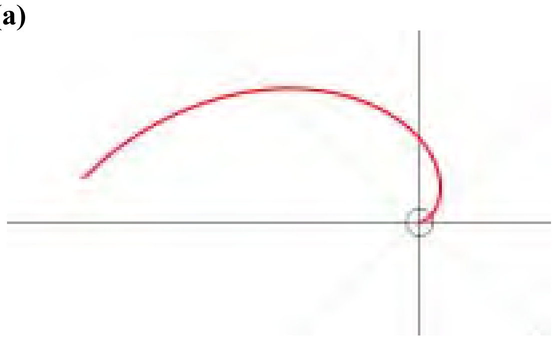
6	(i)	Possible orders are 1, 2, 3, 4, 6 & 12	B1	
		By *Lagrange's Theorem*, the order of an element divides the order of the group (since the order of an element ≡ the order of the subgroup generated by that element)	B1	[2]
	(ii)	E.g. $y = xyx \Rightarrow y\,.\,x^2y = xyx\,.\,x^2y$ by ③	M1	
		$= xy\,.\,x^3\,.\,y = xy\,.\,y^2\,.\,y$ by ②	M1	
		$= x\,.\,y^4 = x\,.\,(y^2)^2$ [by ②]		
		$= x\,.\,(x^3)^2 = x\,.\,e$ by ①		
		2 **M**'s for first, correct uses of 2 different conditions; the **A** for the 3rd condition used to clinch the result.	A1	
		SPECIAL CASE Allow 2/3 for those who correctly argue the converse		[3]
	(iii)	Proving G not abelian: [e.g. by $xyx = y$ but $x^2 \neq e$] $\Rightarrow$ G not cyclic **OR** establishing a contradiction	B1 B1	[2]
7	(i)	$\cos 4\theta + \mathrm{i}\sin 4\theta = (c + \mathrm{i}s)^4$ Use of *de Moivre's Theorem*	M1	
		$= c^4 + 4c^3.\mathrm{i}s + 6c^2.\mathrm{i}^2s^2 + 4c.\mathrm{i}^3s^3 + \mathrm{i}^4s^4$ Binomial expansion attempted	M1	
		$\cos 4\theta = c^4 - 6c^2s^2 + s^4$ and $\sin 4\theta = 4c^3s - 4cs^3$ Equating Re & Im parts	M1	
		$\tan 4\theta = \dfrac{\sin 4\theta}{\cos 4\theta} = \dfrac{4c^3s - 4cs^3}{c^4 - 6c^2s^2 + s^4}$	M1	
		Dividing throughout by c^4 to get $\dfrac{4t - 4t^3}{1 - 6t^2 + t^4}$ legitimately	A1	[5]
	(ii)	$t = \frac{1}{5} \Rightarrow \tan 4\theta = \frac{120}{119}$	B1	
		$\tan\left(\frac{1}{4}\pi + \tan^{-1}\frac{1}{239}\right) = \dfrac{1 + \frac{1}{239}}{1 - \frac{1}{239}} = \frac{120}{119}$	M1 A1	
		Noting that this is $\tan(4\tan^{-1}\frac{1}{5})$ so that $4\tan^{-1}\frac{1}{5} = \frac{1}{4}\pi + \tan^{-1}\frac{1}{239}$	A1	[4]

8	**(i)**	Subst[g]. $x = 1$, $f(1) = 2$ and $f'(1) = 3$ into **(*)** $\Rightarrow$ $f''(1) = 5$	M1 A1	**[2]**
	(ii)	$\{x^2 f'''(x) + 2xf''(x)\} + \{(2x-1)f''(x) + 2f'(x)\} - 2f'(x) = 3e^{x-1}$ Product Rule used twice; at least one bracket correct	M1 A1	
		Subst[g]. $x = 1$, $f'(1) = 3$ and $f''(1) = 5$ into this $\Rightarrow$ $f'''(1) = -12$ **ft** their $f''(1)$	M1 A1	**[4]**
	(iii)	$f(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 + \frac{1}{6}f'''(1)(x-1)^3 + \ldots$ Use of the Taylor series	M1	
		$= 2 + 3(x-1) + \frac{5}{2}(x-1)^2 - 2(x-1)^3 + \ldots$ 1st two terms **cao**; 2nd two terms **ft (i) & (ii)'s answers**	A1 A1	**[3]**
	(iv)	Subst[g]. $x = 1.1$ $\Rightarrow$ $f(1.1) \approx 2.323$ to 3d.p. **cso** (i.e. exactly this answer)	M1 A1	**[2]**
9	**(i)**	$\frac{dy}{dx} + y = 3xy^4$ is a *Bernouilli (differential) equation*		
		$u = \frac{1}{y^3} \Rightarrow \frac{du}{dx} = -\frac{3}{y^4} \times \frac{dy}{dx}$	B1	
		Then $\frac{dy}{dx} + y = 3xy^4$ becomes $-\frac{3}{y^4} \times \frac{dy}{dx} - \frac{3}{y^3} = -9x \Rightarrow \frac{du}{dx} - 3u = -9x$ **AG**	M1 A1	**[3]**
	(ii)	**Method 1**		
		IF is e^{-3x}	M1 A1	
		$\Rightarrow ue^{-3x} = \int -9xe^{-3x}dx$	M1	
		$= 3xe^{-3x} - \int 3e^{-3x}dx$ Use of "parts"	M1	
		$= (3x+1)e^{-3x} + C$	A1	
		Gen. Soln. is $u = 3x + 1 + Ce^{3x}$ **ft**	B1	
		$\Rightarrow y^3 = \frac{1}{3x+1+Ce^{3x}}$ **ft**	B1	
		Using $x = 0, y = \frac{1}{2}$ to find C $C = 7$ or $y^3 = \frac{1}{3x+1+7e^{3x}}$	M1 A1	**[9]**

		Method 2		
		Aux. Eqn. $m - 3 = 0 \Rightarrow u_C = Ae^{3x}$ is the Comp. Fn.	M1 A1	
		For Part. Intgl. try $u_P = ax + b$, $u_P{}' = a$	M1	
		Subst[g]. $u_P = ax + b$ and $u_P{}' = a$ into the d.e. and comparing terms	M1	
		$a - 3ax - 3b = -9x \Rightarrow a = 3, b = 1$ i.e. $u_P = 3x + 1$	A1	
		Gen. Soln. is $u = 3x + 1\ Ae^{3x}$ **ft** PI + CF provided PI has no arbitrary constants and CF has one	B1	
		$\Rightarrow y^3 = \dfrac{1}{3x + 1 + Ae^{3x}}$ **ft**	B1	
		Using $x = 0, y = \dfrac{1}{2}$ to find A $A = 7$ or $y^3 = \dfrac{1}{3x + 1 + 7e^{3x}}$	M1 A1	[9]
10	(i)	Subst[g]. $\begin{pmatrix} 1+3\lambda \\ -3+4\lambda \\ 2+6\lambda \end{pmatrix}$ into plane equation; i.e. $\begin{pmatrix} 1+3\lambda \\ -3+4\lambda \\ 2+6\lambda \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix} = k$ **OR** any point on line (since "given")	M1	
		$k = 2 + 6\lambda + 18 - 24\lambda + 6 + 18\lambda = 26$	A1	[2]
	(ii)	Working with vector $\begin{pmatrix} 10+2m \\ 2-6m \\ 3m-43 \end{pmatrix}$.	B1	
		Subst[g]. into the plane equation: $\begin{pmatrix} 10+2m \\ 2+6m \\ 3m-43 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix} = k$	M1	
		Solving a linear equation in m: $20 + 4m - 12 + 36m + 9m - 129 = 26$	M1	
		$m = 3 \Rightarrow Q = (16, -16, -34)$	A1	
		Sh. Dist. is $\lvert m \rvert \left\lVert \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix} \right\rVert = 21$ or $PQ = \sqrt{6^2 + 18^2 + 9^2} = 21$	A1	
		Alternate methods that find only Sh. Dist. but not Q can score **M1 A1** only		[5]

	(iii)	Finding 3 points in the plane: e.g. $A(1, -3, 2)$, $B(4, 1, 8)$, $C(10, 2, -43)$	M1	
		Then 2 vectors in (// to) plane: e.g. $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 9 \\ 5 \\ -45 \end{pmatrix}$, $\overrightarrow{BC} = \begin{pmatrix} 6 \\ 1 \\ -51 \end{pmatrix}$ **OR B1 B1** for any two vectors in the plane	M1	
		Vector product of any two of these to get normal to plane: $\begin{pmatrix} 10 \\ -9 \\ 1 \end{pmatrix}$ (any non-zero multiple)	M1 A1	
		$d = \begin{pmatrix} 10 \\ -9 \\ 1 \end{pmatrix} \bullet$ (any position vector) $= \begin{pmatrix} 10 \\ -9 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ e.g. $= 39$	M1 A1	
		$\Rightarrow\ 10x - 9y + z = 39$ **cao (or any correct equivalent form)**		**[6]**
		ALTERNATE SOLUTION		
		$ax + by + cz = d$ contains $\begin{pmatrix} 1+3\lambda \\ -3+4\lambda \\ 2+6\lambda \end{pmatrix}$ and $\begin{pmatrix} 10 \\ 2 \\ -43 \end{pmatrix}$		
		... so $a + 3a\lambda + 4b\lambda - 3b + 2c + 6c\lambda = d$ and $10a + 2b - 43c = d$	M1 B1	
		Then $a - 3b + 2c = d$ and $3a + 4b + 6c = 0$ (λ terms) i.e. equating terms	M1	
		Eliminating (e.g.) c from 1st two eqns. $\Rightarrow\ 9a + 10b = 0$	M1	
		Choosing $a = 10, b = -9 \Rightarrow c = 1$ and $d = 39$ i.e. $10x - 9y + z = 39$ **cao**	M1 A1	**[6]**

11	**(i)**	$\lvert w \rvert = \sqrt{\left(\sqrt{3}-1\right)^2 + \left(\sqrt{3}+1\right)} = \sqrt{4-2\sqrt{3}+4+2\sqrt{3}} = \sqrt{8}$ or $2\sqrt{2}$	M1 A1	
		$\arg(w) = \tan^{-1}\left(\frac{\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}\right) = \tan^{-1}\left(2+\sqrt{3}\right) = \frac{5}{12}\pi$	M1 A1	**[4]**
	(ii)	**(a)** $z^3 = \left(2\sqrt{2},\ \frac{5}{12}\pi\right),\ \left(2\sqrt{2},\ \frac{29}{12}\pi\right),\ \left(2\sqrt{2},\ -\frac{19}{12}\pi \text{ or } \frac{53}{12}\pi\right)$		
		$\sqrt[3]{\lvert w \rvert}$; $\frac{\arg(w)}{3}$ These method marks can be earned for just the first root	M1M1	
		$\Rightarrow z = \left(\sqrt{2},\ \frac{5}{36}\pi\right),\ \left(\sqrt{2},\ \frac{29}{36}\pi\right),\ \left(\sqrt{2},\ -\frac{19}{36}\pi\right)$ **A** marks for the 2nd & 3rd roots: r e^(iθ) forms equally acceptable	A1 A1	**[4]**
		(b) z_1, z_2, z_3 the roots of $z^3 - 0.z^2 + 0.z - w = 0$	M1	
		$\Rightarrow z_1 z_2 z_3 = w = \left(\sqrt{3}-1\right) + \mathrm{i}\left(\sqrt{3}+1\right)$	A1	
		ALT. Multiplying the 3 roots together in any form		**[2]**
		(c) 		
		Three points in approx. correct places	M1	
		All equally spaced around a circle, centre O, radius $\sqrt{2}$	M1	
		(Explained that Δ_1 equilateral)	A1	
		$l = 2 \times \sqrt{2}\sin\left(\frac{1}{2} \times \frac{2}{3}\pi\right) = \sqrt{6}$	M1	
		or by the *Cosine Rule*	A1	**[5]**
		(d) $k = \exp\left\{-\mathrm{i}.\frac{5}{36}\pi\right\}$ or $\exp\left\{-\mathrm{i}.\frac{29}{36}\pi\right\}$ or $\exp\left\{\mathrm{i}.\frac{19}{36}\pi\right\}$	B1	**[1]**

12	**(i)**	$I_n = \int_0^3 x^{n-1}\left(x\sqrt{16+x^2}\right)\,dx$ Correct splitting *and* use of parts	M1	
		$= \left[x^{n-1}.\frac{(16+x^2)^{3/2}}{3}\right]_0^3 - \int_0^3 (n-1)x^{n-2}\frac{(16+x^2)^{3/2}}{3}\,dx$	A1	
		$= 3^{n-2}\cdot 125 - \left(\frac{n-1}{3}\right)\int_0^3 x^{n-2}(16+x^2)\sqrt{16+x^2}\,dx$ Method to get 2nd integral of correct form	M1	
		$= 3^{n-2}.125 - \left(\frac{n-1}{3}\right)\{16I_{n-2} + I_n\}$ [i.e. reverting to *I*'s in 2nd integral **ft**]	M1	
		$\Rightarrow 3I_n = 3^{n-1}.125 - 16(n-1)I_{n-2} - (n-1)I_n$ Collecting up I_n's	M1	
		$(n+2)I_n = 125 \times 3^{n-1} - 16(n-1)I_{n-2}$ **AG**	A1	**[6]**
	(ii)	**(a)**  Spiral (with *r* increasing)	B1	
		From *O* to just short of $\theta = \pi$	B1	**[2]**
		(b) $r = \frac{1}{4}\theta^4 \Rightarrow \frac{dr}{d\theta} = \theta^3$ and $r^2 + \left(\frac{dr}{d\theta}\right)^2 = \frac{1}{16}\theta^8 + \theta^6$	M1 A1	
		$L = \int_0^3 \frac{1}{4}\theta^3\sqrt{16+\theta^2}\ \left(= \frac{1}{4}I_3\right)$	M1 A1	
		Now $I_1 = \left[\frac{1}{3}(16+x^2)^{3/2}\right]_0^3 = \frac{61}{3}$	B1	
		and $5I_3 = 125\times 9 - 16\times 2\left(\frac{61}{3}\right) = \frac{1423}{3}$ or $474\frac{1}{3}$ Use of given reduction formula	M1	
		so that $L = \frac{1}{20}\times\frac{1423}{3} = \frac{1423}{60}$ or $23\frac{43}{60}$ or awrt 23.7 **ft** only from suitable $k\,I_3$	A1	
		NB The last 3 marks can be earned by integrating in a variety ways		**[7]**

13		Base-line case: for $n = 5$, 13579 R_5 = 1508 7 6269 contains a string of (5 – 4 = 1) 7's	B1	
		13579 R_6 = 1508776269, 13579 R_7 = 15087776269, etc. or form of 1st & last 4 digits	B1	
		Assume that, for some $k \geq 5$, 13579 R_k = 1508 $\underbrace{77\ldots7}_{(k-4)\text{ 7's}}$ 6269. Induction hypothesis	M1	
		Then, for $n = k + 1$, 13579 R_{k+1} = 13579($10R_k + 1$)	M1	
		Give the **M** mark for the key observation that $R_{k+1} = 10R_k + 1$ **or** $10^k + R_k$, even if not subsequently used.		
		= 1508 $\underbrace{77\ldots7}_{(k-4)\text{ 7's}}$ 62690 + 13579 = 1508 $\underbrace{77\ldots7}_{(k-4+1)\text{ 7's}}$ 76269	A1	
		which contains a string of (k – 4 + 1) 7's, as required. Proof follows by induction (usual round-up).	E1	**[6]**