Cambridge Pre-U

UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
Cambridge International Level 3 Pre-U Certificate
Principal Subject

FURTHER MATHEMATICS **9795/02**

Paper 2 Further Applications of Mathematics **May/June 2012**

3 hours

Additional Materials: Answer Booklet/Paper
Graph Paper
List of Formulae (MF20)

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READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
Where a numerical value for the acceleration due to gravity is needed, use $10\,\mathrm{m\,s^{-2}}$.
The use of an electronic calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 120.

This document consists of **6** printed pages and **2** blank pages.

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UNIVERSITY *of* CAMBRIDGE International Examinations

Section A: Probability (60 marks)

1 The random variable X has probability density function $\mathrm{f}(x)$, where

$$\mathrm{f}(x) = \begin{cases} k\mathrm{e}^{-kx} & x \geqslant 0, \\ 0 & x < 0, \end{cases}$$

and k is a positive constant.

(i) Show that the moment generating function of X is $\mathrm{M}_X(t) = k(k-t)^{-1}, \quad t < k$. [3]

(ii) Use the moment generating function to find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [5]

2 The independent random variables X and Y have normal distributions where $X \sim \mathrm{N}(\mu, \sigma^2)$ and $Y \sim \mathrm{N}(3\mu, 4\sigma^2)$. Two random samples each of size n are taken, one from each of these normal populations.

(i) Show that $a\overline{X} + b\overline{Y}$ is an unbiased estimator of μ provided that $a + 3b = 1$, where a and b are constants and $\overline{X}$ and $\overline{Y}$ are the respective sample means. [3]

In the remainder of the question assume that $a\overline{X} + b\overline{Y}$ is an unbiased estimator of μ.

(ii) Show that $\mathrm{Var}\left(a\overline{X} + b\overline{Y}\right)$ can be written as $\dfrac{\sigma^2}{n}(1 - 6b + 13b^2)$. [3]

(iii) The value of the constant b can be varied. Find the value of b that gives the minimum of $\mathrm{Var}\left(a\overline{X} + b\overline{Y}\right)$, and hence find the minimum of $\mathrm{Var}\left(a\overline{X} + b\overline{Y}\right)$ in terms of σ and n. [3]

3 Small amounts of a potentially hazardous chemical are discharged into a river from a nearby industrial site. A random sample of size 6 was taken from the river and the concentration of the chemical present in each item was measured in grams per litre. The results are shown below.

1.64 1.53 1.78 1.60 1.73 1.77

(i) Assuming that the sample was taken from a normal distribution with known variance 0.01, construct a 99% confidence interval for the mean concentration of the chemical present in the river. [4]

(ii) If instead the sample was taken from a normal distribution, but with unknown variance, construct a revised 99% confidence interval for the mean concentration of the chemical present in the river. [5]

(iii) If the mean concentration of the chemical in the river exceeds 1.8 grams per litre, then remedial action needs to be taken. Comment briefly on the need for remedial action in the light of the results in parts **(i)** and **(ii)**. [1]

4 **(i)** The random variable X has the distribution $\text{Po}(\lambda)$. Prove that the probability generating function, $\text{G}_X(t)$, is given by

$$\text{G}_X(t) = \text{e}^{\lambda(t-1)}. \quad [3]$$

(ii) The independent random variables X and Y have distributions $\text{Po}(\lambda)$ and $\text{Po}(\mu)$ respectively. Use probability generating functions to show that the distribution of $X + Y$ is $\text{Po}(\lambda + \mu)$. [2]

(iii) Given that $X \sim \text{Po}(1.5)$ and $Y \sim \text{Po}(2.5)$, find $\text{P}(X \leqslant 2 \mid X + Y = 4)$. [5]

5 **(i)** The probability that a shopper obtains a parking space on the river embankment on any given Saturday morning is 0.2. Using a suitable normal approximation, find the probability that, over a period of 100 Saturday mornings, the shopper finds a parking space

(a) at least 15 times,

(b) no more than 12 times. [6]

(ii) The number of parking tickets that a traffic warden issues on the river embankment during the course of a week has a Poisson distribution with mean 36. The probability that the traffic warden issues more than N parking tickets is less than 0.05. Using a suitable normal approximation, find the least possible value of N. [5]

6 The lengths of time, in years, that sales representatives for a certain company keep their company cars may be modelled by the distribution with probability density function $\text{f}(x)$, where

$$\text{f}(x) = \begin{cases} \frac{4}{27}x^2(3 - x) & 0 \leqslant x \leqslant 3, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Draw a sketch of this probability density function. [2]

(ii) Calculate the mean and the mode of X. [6]

(iii) Comment briefly on the values obtained in part **(ii)** in relation to the sketch in part **(i)**. [1]

(iv) Given that $\sigma^2 = 0.36$, find $\text{P}\big(|X - \mu| < \sigma\big)$, where μ and σ^2 denote the mean and the variance of X respectively. [3]

Section B: Mechanics (60 marks)

7 A cyclist and her machine have a combined mass of 90 kg and she is riding along a straight horizontal road. She is working at a constant power of 75 W. At time t seconds her speed is $v\,\text{m}\,\text{s}^{-1}$ and the resistance to motion is kv N, where k is a constant.

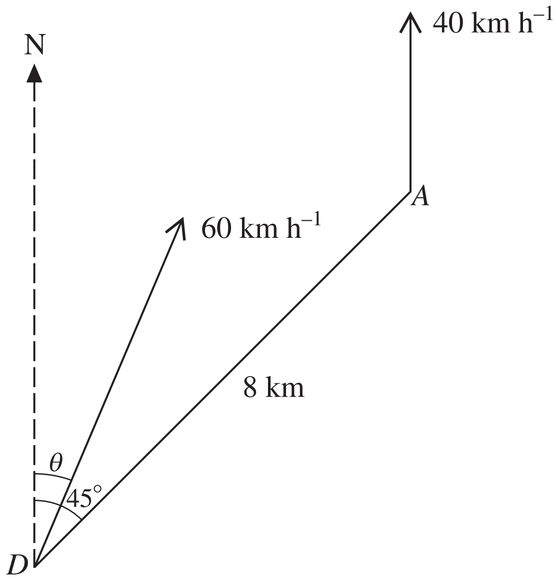
(i) If the cyclist's maximum steady speed is $10\,\text{m}\,\text{s}^{-1}$, show that $k = \frac{3}{4}$. [1]

(ii) Use Newton's second law to show that

$$\frac{25}{v} - \frac{v}{4} = 30\frac{\text{d}v}{\text{d}t}. \quad [2]$$

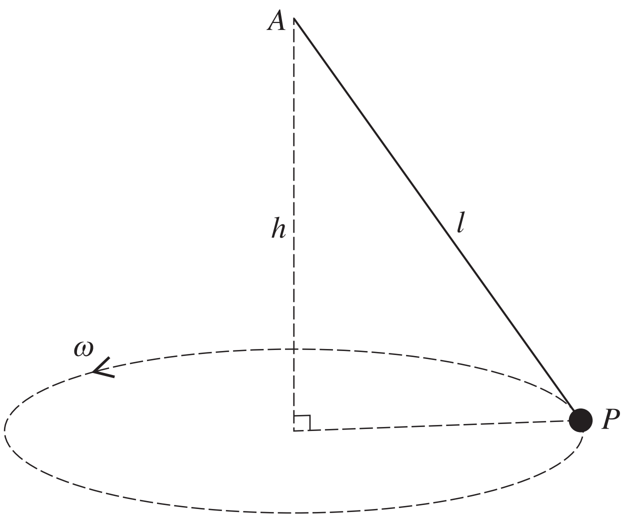
(iii) Find the time taken for the cyclist to accelerate from a speed of $3\,\text{m}\,\text{s}^{-1}$ to a speed of $7\,\text{m}\,\text{s}^{-1}$. [5]

8



An aircraft carrier, A, is heading due north at $40\,\text{km}\,\text{h}^{-1}$. A destroyer, D, which is 8 km south-west of A, is ordered to take up a position 3 km east of A as quickly as possible. The speed of D is $60\,\text{km}\,\text{h}^{-1}$ (see diagram). Find the bearing, θ, of the course that D should take, giving your answer to the nearest degree. [8]

9



A particle P of mass m is attached to one end of a light inextensible string of length l. The other end of the string is attached to a fixed point A. The particle moves with constant angular speed ω in a horizontal circle whose centre is at a distance h vertically below A (see diagram).

(i) Show that however fast the particle travels AP will never become horizontal, and that the tension in the string is always greater than the weight of the particle. [3]

(ii) Find the tension in the string in terms of m, l and ω. [2]

(iii) Show that $\omega^2 h = g$ and calculate ω when h is 0.5 m. [4]

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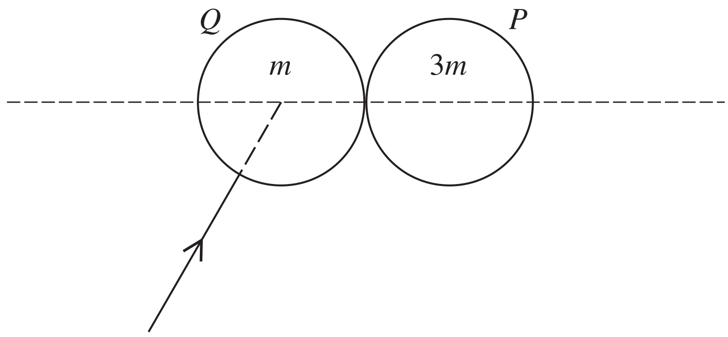

A smooth sphere P of mass $3m$ is at rest on a smooth horizontal table. A second smooth sphere Q of mass m and the same radius as P is moving along the table towards P and strikes it obliquely (see diagram). After the collision, the directions of motion of the two spheres are perpendicular.

(i) Find the coefficient of restitution. [5]

(ii) Given that one sixth of the original kinetic energy is lost as a result of the collision, find the angle between the initial direction of motion of Q and the line of centres. [7]

11 Two light strings, each of natural length a and modulus of elasticity $6mg$, are attached at their ends to a particle P of mass m. The other ends of the strings are attached to two fixed points A and B, which are at a distance $6a$ apart on a smooth horizontal table. Initially P is at rest at the mid-point of AB. The particle is now given a horizontal impulse in the direction perpendicular to AB. At time t the displacement of P from the line AB is x.

(i) Show that

$$\ddot{x} = -\frac{12gx}{a}\left(1 - \frac{a}{\sqrt{9a^2 + x^2}}\right).$$ [6]

(ii) Given that $\frac{x}{a}$ is small throughout the motion, show that the equation of motion is approximately

$$\ddot{x} = -\frac{8gx}{a}$$

and state the period of the simple harmonic motion that this equation represents. [3]

(iii) Given that the initial speed of P is $\sqrt{\frac{ga}{200}}$, show that the amplitude of the simple harmonic motion is $\frac{1}{40}a$. [2]

12 A projectile is launched from the origin with speed $20\,\mathrm{m\,s^{-1}}$ at an angle α above the horizontal.

(i) Prove that the equation of its trajectory is

$$y = x\tan\alpha - \frac{x^2}{80}(1 + \tan^2\alpha).$$ [4]

(ii) Regarding the equation of the trajectory as a quadratic equation in $\tan\alpha$, show that $\tan\alpha$ has real values provided that

$$y \leqslant 20 - \frac{x^2}{80}.$$ [4]

(iii) A plane is inclined at an angle β to the horizontal. The line l, with equation $y = x\tan\beta$, is a line of greatest slope in the plane. A particle is projected from a point on the plane, in the vertical plane containing l. By considering the intersection of l with the bounding parabola $y = 20 - \frac{x^2}{80}$, deduce that the maximum range up, or down, this inclined plane is $\frac{40}{1+\sin\beta}$, or $\frac{40}{1-\sin\beta}$, respectively. [4]

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