Cambridge Pre-U

Cambridge International Examinations
Cambridge Pre-U Certificate

FURTHER MATHEMATICS **9795/01**

Paper 1 Further Pure Mathematics **May/June 2016**

MARK SCHEME

Maximum Mark: 120

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge will not enter into discussions about these mark schemes.

Cambridge is publishing the mark schemes for the May/June 2016 series for most Cambridge IGCSE®, Cambridge International A and AS Level components and some Cambridge O Level components.

CAMBRIDGE
International Examinations

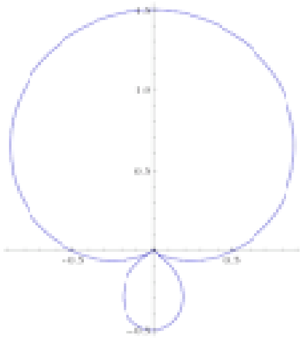
Question	Answer	Marks	Notes
1	$\sum_{r=1}^{n}\left(8r^3+r\right)\equiv 8\sum_{r=1}^{n} r^3+\sum_{r=1}^{n} r$	M1	Splitting into separate series
	$\equiv 8\times\frac{1}{4}n^2(n+1)^2+\frac{1}{2}n(n+1)$	M1 M1	Both used good factorisation attempt
	$\equiv \frac{1}{2}n(n+1)\left\{4n^2+4n+1\right\}$		
	$\equiv \frac{1}{2}n(n+1)(2n+1)^2$	A1 [4]	Legitimate **(AG)**
2	$\begin{pmatrix}6\\2\\5\end{pmatrix}\times\begin{pmatrix}-6\\1\\4\end{pmatrix}=3\begin{pmatrix}1\\-18\\6\end{pmatrix}$	M1 A1	Attempt at vector products of the d.v.s (any suitable multiple)
	Shortest Distance = $\left\| (\mathbf{b}-\mathbf{a}) \bullet \hat{\mathbf{n}} \right\|$	M1	
	$=\frac{1}{19}\begin{pmatrix}10\\-2\\5\end{pmatrix}\bullet\begin{pmatrix}1\\-18\\6\end{pmatrix}=\frac{1}{19}\left(10+36+30\right)$	B1 B1	$\left\|\hat{\mathbf{n}}\right\|$ correct Sc. Prod. **ft** correct
	$=4$	A1 [6]	
	Alternative method: **M1 A1** for common normal $\mathbf{i}-18\mathbf{j}+6\mathbf{k}$ **M1 A1** for parallel planes $x-18y+6z=-55$ and -131 **M1 A1** for Sh.D formula, $\frac{\|131-55\|}{\|\mathbf{n}\|}=\frac{76}{19}=4$		
3 (i)	$\frac{2x^2-x-1}{2x-3}=k \Rightarrow 2x^2-(2k+1)x+(3k-1)=0$	B1	**(AG)** Shown legitimately
	For non-real x, $(2k+1)^2-8(3k-1)<0$	M1	Considering discriminant (or equivalent)
	$4k^2-20k+9<0 \Rightarrow (2k-1)(2k-9)<0$	M1	Solving from $\Delta<0$
	$\Rightarrow$ no curve for $\frac{1}{2}<k=y<\frac{9}{2}$	A1 [4]	**(AG)** Must be satisfactorily explained
(ii)	TPs at $y=\frac{1}{2}$ $\quad$ $y=\frac{9}{2}$	M1	First y (k) substituted back
	i.e. $2x^2-2x+\frac{1}{2}=0$ $\quad$ $2x^2-10x+\frac{25}{2}=0$	M1	Second y (k) substituted back
	$x=\frac{1}{2}$ $\quad$ $x=\frac{5}{2}$	A1A1 [4]	
	<u>Alternative method:</u> when $\Delta=0$, **M1** $x=$ "$-\frac{b}{2a}$" $=\frac{2k+1}{4}$ **M1** $\Rightarrow x=\frac{1}{2}$ $(y=\frac{1}{2})$ & $x=\frac{5}{2}$ $(y=\frac{9}{2})$ **A1 A1** **Note:** For finding TPs via $\frac{dy}{dx}=0$, max. **M1 A1** since qn. asks for a "deduce" method		

Question	Answer	Marks	Notes
4 (i)	Attempt at det(**M**) Det = 0 *Shown*	M1 A1 [2]	 (Or via full alternative algebraic method)
(ii)	$-x+3y+z=1$ $5x-y+2z=16$ $-x+y \quad =-2$ parametrisation attempt (or equivalent) started: e.g. set $x=\lambda$, then $y=\lambda-2$ complete attempt: $z = 1 + \lambda - 3\lambda + 6 = 7 - 2\lambda$ all correct (p.v. and d.v.) … may be in vector line eqn. form: $\mathbf{r} = \begin{pmatrix}0\\-2\\7\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\-2\end{pmatrix}$ <u>Alternative method 1:</u> **B1** as above, followed by (e.g.): Finding two distinct points on the solution line; e.g. (2, 0, 3), (0, –2, 7) **M1 A1** Then eqn. of line containing these 2 points **M1 A1** possibly **ft** for line (of intersection) of 3 planes (given by the 3 eqns.) **B1** <u>Alternative method 2:</u> **B1** as above, followed by: Vector product of any two plane normals **M1A1** Finding coords. or p.v. of any pt. on line **B1** Eqn. of line using these results appropriately **B1** for line (of intersection) of 3 planes (given by the 3 eqns.) **B1**	 B1 M1 M1 A1A1 [6]	 for all three
5	Aux. Eqn. $m^2 - 4m + 5 = 0$ $m = 2 \pm \mathrm{i}$ Comp. Fn. is $y_C = \mathrm{e}^{2x}\left(A\cos x + B\sin x\right)$ For Part. Intgl. try $y = y_p = a\,\mathrm{e}^{2x}$ Both $y' = 2a\,\mathrm{e}^{2x}$ and $y'' = 4a\,\mathrm{e}^{2x}$ Subst$^{\text{g}}$. into given d.e. & solving to find a : $y_p = 24\mathrm{e}^{2x}$ Gen. Soln. $y = \mathrm{e}^{2x}\left(A\cos x + B\sin x + 24\right)$	M1 A1 B1ft B1 B1 M1 A1 B1ft [8]	Including solving attempt $\left(4a - 8a + 5a\right)\mathrm{e}^{2x} = 24\mathrm{e}^{2x}$ $y_C + y_P$ provided y_C has 2 arbitrary constants and y_P has none. Also, A, B must be real here

Question	Answer	Marks	Notes
6 (i)	For f(x) = sinh x + sin x – 3x,		
	f(2.5) = – 0.851… < 0 and f(3) = 1.159… > 0	M1	or LHS < RHS and then LHS > RHS
	Change-of-sign (for a continuous fn.)		
	$\Rightarrow$ 2.5 < α < 3	A1 [2]	All correctly shown/explained
(ii)	$\sinh x + \sin x = \left(x + \frac{x^3}{3\,!} + \frac{x^5}{5\,!} + \frac{x^7}{7\,!} + \frac{x^9}{9\,!} + \ldots\right) + \left(x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \frac{x^7}{7\,!} + \frac{x^9}{9\,!} - \ldots\right)$	M1	for use of both series (attempted)
	$= 2x + \frac{x^5}{60} + \ldots$	A1	
	$2x + \frac{x^5}{60} = 3x \quad \Rightarrow \; (x \neq 0)\; x^4 = 60$		
	$\Rightarrow \; \alpha \approx \sqrt[4]{60}$ (2.783 158 …)	B1 [3]	**(AG)** shown legitimately
(iii)	Using $2x + \frac{x^5}{60} + \frac{x^9}{181\,440} = 3x$ with $x \neq 0$	M1	
	Solving as a quadratic in x^4	M1	$x^8 + 3024x^4 - 181\,440 = 0$
	$\alpha \approx 2.769\,8$ (to 4 d.p.)	A1	from $x^4 = \sqrt{2\,467\,584} - 1512$, $x = \sqrt[4]{58.854\,5\ldots}$
	[c.f. actual root 2.769 7 to 4 d.p.]	[3]	
7 (i)	$\lvert z^3 \rvert = 2\sqrt{2}$ arg(z^3) = $\frac{1}{4}\pi$	B1B1	
	$\Rightarrow z = \left(\sqrt{2},\; \frac{1}{12}\pi\right)$ cube-rooting modulus; arg ÷3	M1M1	(in at least the first case)
	Other two roots: $\left(\sqrt{2},\; \frac{3}{4}\pi\right)$ and $\left(\sqrt{2},\; \frac{17}{12}\pi\right)$	A1A1 [6]	
(ii)	Equilateral Δ with vertices in approx. correct places	B1	
	Area = $3 \times \frac{1}{2} \times \sqrt{2} \times \sqrt{2} \sin\left(\frac{2}{3}\pi\right) = \frac{3}{2}\sqrt{3}$	M1A1	Give M1 for any correct area
	Accept **awrt** 2.60 (3 s.f.) from correct working	[3]	

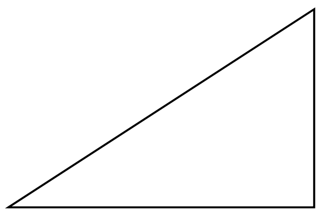
Question	Answer	Marks	Notes
8 (i) (a)	**G** | **1** | **2** | **4** | **8** | **16** | **32**		
	1 | 1 | 2 | 4 | 8 | 16 | 32		
	2 | 2 | 4 | 8 | 16 | 32 | 1	M1	for mostly correct
	4 | 4 | 8 | 16 | 32 | 1 | 2	A1	for all correct
	8 | 8 | 16 | 32 | 1 | 2 | 4		
	16 | 16 | 32 | 1 | 2 | 4 | 8		
	32 | 32 | 1 | 2 | 4 | 8 | 16	[2]	
(b)	$(S, \times_{63})$ closed, since no new elements in table	B1	
	$\times_{63}$ is associative (given)		
	1 is the identity element	B1	
	Each (non-identity) element has a unique inverse: $2 \leftrightarrow 32$, $4 \leftrightarrow 16$ and 8 is self-inverse	B1 [3]	All must be identified
(ii) (a)	**H** | **e** | **x** | **y** | **y^2** | **xy** | **yx**		
	e | e | x | y | y^2 | xy | yx	B1	for last 3 elements (any forms)
	x | x | e | xy | yx | y | y^2	B1	for identity row/column (green)
	y | y | yx | y^2 | e | x | xy	B1	for easy elements (gold) or $\geqslant$ 14 others
	y^2 | y^2 | xy | e | y | yx | x	B1	for all
	xy | xy | y^2 | yx | x | e | y		
	yx | yx | y | x | xy | y^2 | e	[4]	
(b)	Proper subgroups of H are (condone inclusion of $\{e\}$ and H): $\{e, x\}$, $\{e, xy\}$, $\{e, yx\}$ and $\{e, y, y^2\}$	B1B1 [2]	B1 Any 2; +B1 all 4 and no extras
(c)	G and H are NOT isomorphic e.g. Different numbers of self-inverse elements / elements of order 3 or G cyclic, H non-cyclic or G abelian, H non-abelian	B1 [1]	Correct conclusion WITH a valid reason

Question	Answer	Marks	Notes
9 (i)	$\alpha+\beta+\gamma=a$, $\alpha\beta+\beta\gamma+\gamma\alpha=b$ and $\alpha\beta\gamma=c$	B1B1 [2]	B1 any 2 correct; + B1 all 3 correct
(ii)	$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$	M1	
	$=a^2-2b$	A1	
	$\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2=(\alpha\beta+\beta\gamma+\gamma\alpha)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma)$	M1	
	$=b^2-2ac$	A1 [4]	
(iii)	$(\alpha-2\beta\gamma)(\beta-2\gamma\alpha)(\gamma-2\alpha\beta)$		
	$=\left(\alpha\beta-2\beta^2\gamma-2\alpha^2\gamma+4\gamma^2\alpha\beta\right)(\gamma-2\alpha\beta)$	M1	
	$=\alpha\beta\gamma-2(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2)+4\alpha\beta\gamma(\alpha^2+\beta^2+\gamma^2)-8(\alpha\beta\gamma)^2$	M1	Collecting up in terms of the symmetric fns.
	$=c-2\left(b^2-2ac\right)+4c\left(a^2-2b\right)-8c^2$	M1	Use of **(i)**'s and **(ii)**'s results
	$=c\left(1+4a+4a^2\right)-2\left(b^2+4bc+4c^2\right)$		
	$=c(2a+1)^2-2(b+2c)^2$	A1 [4]	**legitimately**
	Alternative method:		
	Using $\alpha\beta\gamma=c$,		
	$(\alpha-2\beta\gamma)(\beta-2\gamma\alpha)(\gamma-2\alpha\beta)$		
	$=\left(\alpha-\frac{2c}{\alpha}\right)\left(\beta-\frac{2c}{\beta}\right)\left(\gamma-\frac{2c}{\gamma}\right)$		
	$=\frac{1}{\alpha\beta\gamma}\left(\alpha^2-2c\right)\left(\beta^2-2c\right)\left(\gamma^2-2c\right)=$		
	$\frac{1}{c}\left((\alpha\beta\gamma)^2-2c\sum\alpha^2\beta^2+4c^2\sum\alpha^2-8c^3\right)$		
	$=\frac{1}{c}\left(c^2-2c\left[b^2-2ac\right]+4c^2\left[a^2-2b\right]-8c^3\right)$		
	= etc. as above		
(iv)	One root is the product of the other two		
	$\Leftrightarrow\ (\alpha-2\beta\gamma)(\beta-2\gamma\alpha)(\gamma-2\alpha\beta)=0$		
	$\Leftrightarrow\ c(2a+1)^2=2(b+2c)^2$	B1	**legitimately**
	Must reason $\Rightarrow$ and $\Leftarrow$ explicitly (or together)	[1]	

Question	Answer	Marks	Notes
10 (i)		M1A1	$\frac{1}{2}+\sin\theta=0$ when $\theta=\frac{7}{6}\pi$, $\frac{11}{6}\pi$
		B1	Symmetry in y-axis
		B1	$\left(\frac{1}{2},\ 0\right)$ on initial line
		B1	Correct upper portion
		B1 [6]	Correct lower portion
(ii)	$A=\left(\frac{1}{2}\right)\int_0^{2\pi}\left(\frac{1}{2}+\sin\theta\right)^2\,\mathrm{d}\theta$	M1	Penalise incorrect multiples with final A0
	$=\frac{1}{2}\int_0^{2\pi}\left(\frac{1}{4}+\sin\theta+\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$	M1	Double-angle formula
	$=\frac{1}{2}\left[\frac{3}{4}\theta-\cos\theta-\frac{1}{4}\sin 2\theta\right]_0^{2\pi}$	A1	correctly integrated 3 suitable terms
	$=\frac{3}{4}\pi$	A1 [4]	

Question	Answer	Marks	Notes
11 (i)	$F_3 = 2,\ F_4 = 3,\ F_5 = 5,\ F_6 = 8$	B1 [1]	all
(ii) (a)	$\mathrm{p}_2(x) = 1 + \dfrac{1}{x+1} = \dfrac{x+2}{x+1}$	B1	
	$\mathrm{p}_3(x) = \dfrac{2x+3}{x+2}$	B1	
	$\mathrm{p}_4(x) = \dfrac{3x+5}{2x+3}$	B1 [3]	**(AG))**
(b)	$\mathrm{p}_n(x) = \dfrac{F_n\ x + F_{n+1}}{F_{n-1}\ x + F_n}$	B1	
	Result is true for $n = 2$ (and 3 and 4)	B1	May be mentioned in later in their “round up”
	Assuming $\mathrm{p}_k(x) = \dfrac{F_k\ x + F_{k+1}}{F_{k-1}\ x + F_k}$ (not separate from their conjecture)		
	$\mathrm{p}_{k+1}(x) = 1 + \dfrac{F_{k-1}\ x + F_k}{F_k\ x + F_{k+1}}$	M1	
	$= \dfrac{F_k\ x + F_{k+1}}{F_k\ x + F_{k+1}} + \dfrac{F_{k-1}\ x + F_k}{F_k\ x + F_{k+1}}$		
	$= \dfrac{\left(F_k + F_{k-1}\right)\ x + \left(F_k + F_{k+1}\right)}{F_k\ x + F_{k+1}}$	M1	Collecting coeffts. into successive Fib. terms
	$= \dfrac{F_{k+1}\ x + F_{k+2}}{F_k\ x + F_{k+1}}$	A1	
	which is the required formula with $n = k + 1$. Accept this as sufficient that proof follows by induction.	[5]	

Question	Answer	Marks	Notes
12 (i)	$y=\ln\left(\tanh\frac{1}{2}x\right) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\tanh\frac{1}{2}x}\cdot\frac{1}{2}\mathrm{sech}^2\frac{1}{2}x$	M1A1	
	$=\mathrm{cosech}\,x$	A1 [3]	**(AG)**
(ii) (a)	$L_n=\int_n^{2n}\sqrt{1+\mathrm{cosech}^2x}\,\mathrm{d}x$	M1	
	$=\int_n^{2n}\coth x\,\mathrm{d}x$	A1	
	$=\left[\ln(\sinh x)\right]$	A1	correct integrn.
	$\ln\left(\frac{\sinh 2n}{\sinh n}\right)=\ln\left(\frac{e^{2n}-e^{-2n}}{e^n-e^{-n}}\right)$	M1	
	$\approx\ln\left(\frac{e^{2n}}{e^n}\right)$, for large n, $=\ln\left(e^n\right)=n$	A1 [5]	**legitimately**
	OR $\ln\left(\frac{\sinh 2n}{\sinh n}\right)=\ln\left(2\cosh n\right)=\ln\left(e^n+e^{-n}\right)$ **M1**		
	$\approx\ln\left(e^n\right)$ for large n, $=n$ **A1**		**legitimately**
(b)	Method (sketch or statement) to indicate that C asymptotically "merges" with the x-axis	M1	
	so that C is approximately a horizontal straight-line from $(n, 0)$ to $(2n, 0)$	A1 [2]	
13 (i) (a)	Let $y=\sec^{-1}x$, i.e. $\sec y=x$		
	$\Rightarrow \cos y=\frac{1}{x} \Rightarrow y=\cos^{-1}\left(\frac{1}{x}\right)$	B1	
	Then $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sec^{-1}x\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos^{-1}\frac{1}{x}\right)$		
	$=-\frac{1}{\sqrt{1-(1/x)^2}}\times\frac{-1}{x^2}$	M1	(Using MF20 and the *Chain Rule*)
	$=\frac{1}{x\sqrt{x^2-1}}$	A1 [3]	**(AG)**
	[Allow **M1 A1** for valid non-"deduced" approaches]		

Question	Answer	Marks	Notes
(b)	$\int \sec^{-1} x \,.\, 1 \ \mathrm{d}x$	M1	Use of integration by "parts"
	$= x \,.\, \sec^{-1} x - \int x \,.\, \dfrac{1}{x\sqrt{x^2-1}} \mathrm{d}x$	A1 A1	
	$= \left[x \,.\, \sec^{-1} x - \cosh^{-1} x \right]$	A1 [4]	Condone lack of "+ C"
(ii) (a)	$\dfrac{1}{x\sqrt{x^2-1}} = \dfrac{1}{\sqrt{2}} \Rightarrow x^2\left(x^2-1\right) = 2$		
	$\Rightarrow x^4 - x^2 - 2 = \left(x^2-2\right)\left(x^2+1\right) = 0$	M1	
	$\Rightarrow x = \sqrt{2}$ and $y = \frac{1}{4}\pi$	A1 A1	i.e. $P = (\sqrt{2}, \frac{1}{4}\pi)$
			
	$\dfrac{\frac{1}{4}\pi}{\sqrt{2}-c} = \dfrac{1}{\sqrt{2}}$	M1 A1	or by $y - \frac{1}{4}\pi = \dfrac{1}{\sqrt{2}}\left(x - \sqrt{2}\right)$ & $y = 0$
	$c = \sqrt{2} - \dfrac{\pi\sqrt{2}}{4}$	A1 [6]	i.e. $Q = \left(\sqrt{2} - \dfrac{\pi\sqrt{2}}{4},\ 0\right)$
(b)	Area $\Delta = \dfrac{1}{2} \times \dfrac{\pi\sqrt{2}}{4} \times \dfrac{\pi}{4} = \dfrac{\pi^2\sqrt{2}}{32}$	B1	
	Area under curve = $\sqrt{2} \,.\, \dfrac{\pi}{4} - \ln\left(1+\sqrt{2}\right)$	B1	using **(iii)**'s answer and the limits $(1, \sqrt{2})$
	Then $R = \dfrac{\pi^2\sqrt{2}}{32} - \dfrac{\pi\sqrt{2}}{4} + \ln\left(1+\sqrt{2}\right)$	M1	Difference in areas
	$= \ln\left(1+\sqrt{2}\right) - \dfrac{\pi(8-\pi)\sqrt{2}}{32}$	A1 [4]	**(AG)**